Problem of the Week
Problem E and Solution
All Around the Cube

Problem
A cube is said to be inscribed in a sphere when all the vertices of the cube are on the surface of the sphere. In the diagram below, the cube is inscribed in the sphere with centre $O$. If the radius of the sphere is 6 cm, determine the volume of the cube.

Solution
Label four of the vertices of the cube $A$, $B$, $C$, $D$, as shown in the diagram. Let $x$ represent the side length of the cube. Then $AB = BC = CD = x$.

The diagonals of a cube intersect at a point such that the distance from the intersection point to each vertex is equal. Since each vertex of the cube is on the sphere, the diagonal of the cube, $AD$, is equal in length to the diameter of the sphere. Therefore, $AD = 2(6) = 12$ cm.

Each face of a cube is a square, so $\angle ABC = 90^\circ$. Using the Pythagorean Theorem in $\triangle ABC$,

$$AC^2 = AB^2 + BC^2 = x^2 + x^2 = 2x^2$$

In a cube the sides are perpendicular to the base. In particular, $DC$ is perpendicular to the base and it follows that $DC \perp AC$. Therefore $\triangle DCA$ is a right-angled triangle. Using the Pythagorean Theorem in $\triangle DCA$,

$$AD^2 = AC^2 + CD^2 = 2x^2 + x^2 = 3x^2$$

But $AD = 12$, so $AD^2 = 144$. Then,

$$3x^2 = 144$$

$$x^2 = 48$$

$$x = 4\sqrt{3}, \quad \text{since } x > 0$$

Therefore, the volume of the cube is $x^3 = (4\sqrt{3})^3 = 192\sqrt{3}$ cm$^3$. 