Problem of the Week
Problem E and Solution
All the Children

Problem

My family has four children, each with a different age. The product of their ages is 17 280. The sum of the ages of the three oldest children is 40 and the sum of the ages of the three youngest children is 32. Determine all possibilities for the ages of the four children.

Solution

Let the ages of the children from youngest to oldest be $a,\ b,\ c,\ d$.

Since the ages of the three oldest children sum to 40, $b + c + d = 40$. (1)
Since the ages of the three youngest children sum to 32, $a + b + c = 32$. (2)

Subtracting (2) from (1), we obtain $d - a = 8$. This means that the difference between the age of the oldest child and the age of the youngest child is 8.

Now we can factor 17 280.
$17\,280 = 2^7 \times 3^3 \times 5 = (2^2 \times 3) \times (2^2 \times 3) \times (2^2 \times 3) \times (2 \times 5) = 12 \times 12 \times 12 \times 10$. Since the children all have different ages, we can use this statement to help us figure out the possible ages of the oldest and youngest children.

Is it possible that the oldest child is 12? If the oldest child is 12, then the youngest child would be $12 - 8 = 4$. The largest product that could be generated using two more different ages between 4 and 12 would be $12 \times 11 \times 10 \times 4 = 5280$. Since this is less than 17 280, it follows that the oldest child must be older than 12.

Is it possible that the youngest child is 10? If the youngest child is 10, then the oldest child would be $10 + 8 = 18$. The smallest product that could be generated using two more different ages between 10 and 18 would be $10 \times 11 \times 12 \times 18 = 23\,760$. Since this is greater than 17 280, it follows that the youngest child must be younger than 10.

There are now a limited number of possibilities to consider for the ages of the youngest and oldest children. The possibilities are $(5, 13)$, $(6, 14)$, $(7, 15)$, $(8, 16)$, and $(9, 17)$. No other combinations are possible since the oldest child must be older than 12 and the youngest child must be younger than 10.

The prime numbers 7, 13, and 17 are not factors of 17 280. Therefore we can eliminate the possibilities where an age is one of 7, 13, or 17, leaving $(6, 14)$ and $(8, 16)$. Since 14 is a multiple of 7, we can conclude that it is also not a factor of 17 280. Thus, we can eliminate $(6, 14)$. Now there is only one possibility left to consider, namely $(8, 16)$.

Now $17\,280 = 8 \times 16 \times 3^3 \times 5$. Using the remaining factors 3^3 and 5, we need to create two numbers between 8 and 16. The only possibilities are $3^2 = 9$ and $3 \times 5 = 15$.

Therefore, the only possibility is that the ages of the children are 8, 9, 15, and 16. It is easy to verify that this is a valid solution.