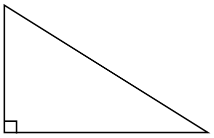


Problem of the Week
Problem E and Solution
That's Right

Problem

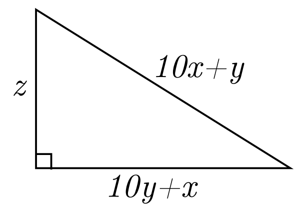
The side lengths of a right-angled triangle are all positive two-digit integers. If the digits representing the length of the hypotenuse are the digits of one of the side lengths written in the reverse order, find all the possible lengths of the hypotenuse.

Solution

Let x represent the tens digit of the hypotenuse such that x is an integer from 1 to 9. Let y represent the units digit of the hypotenuse such that y is an integer from 1 to 9. Then the length of the hypotenuse is $10x + y$.

Since one of the sides has the same digits as the hypotenuse in reverse order, the length of this side is $10y + x$.

Let the third side be z such that z is a two digit integer.



$$\begin{array}{rrcl}
\text{Using the Pythagorean Theorem:} & (10y+x)^2 + z^2 & = & (10x+y)^2 \\
\text{Expanding:} & 100y^2 + 20xy + x^2 + z^2 & = & 100x^2 + 20xy + y^2 \\
\text{Rearranging:} & z^2 & = & 99x^2 - 99y^2 \\
\text{Factoring:} & z^2 & = & 99(x-y)(x+y)
\end{array}$$

Since z^2 is a perfect square, $99(x + y)(x - y)$ must also be a perfect square.
But $99(x + y)(x - y) = 9(11)(x + y)(x - y)$. So, to be a perfect square, $(x + y)(x - y)$ must contain a factor of 11. Since x and y are each integers from 1 to 9, $x - y$ cannot be equal to 11 and $x + y$ cannot be greater than 18, and so we must have $x + y = 11$. Now, for $9(11)(x + y)(x - y) = 9(11)(11)(x - y)$ to be a perfect square, $x - y$ must be a perfect square. Since x and y are integers from 1 to 9, there are three possibilities for $x - y$ that give a perfect square; $x - y = 1$, $x - y = 4$ or $x - y = 9$. These are the three possibilities:
$(x + y)(x - y) = 11(1)$ or $(x + y)(x - y) = 11(4)$ or $(x + y)(x - y) = 11(9)$.

Case 1: $(x + y)(x - y) = 11(1)$
If $x + y = 11$ and $x - y = 1$, we solve the system of equations obtaining $x = 6$ and $y = 5$. This gives a hypotenuse of $10x + y = 65$ and second side $10y + x = 56$. Then solving for z, $z^2 = 99(x + y)(x - y) = 99(11)(1) = 1089$ and $z = 33$. This solution is easily confirmed but is it the only solution?

Case 2: $(x + y)(x - y) = 11(4)$
If $x + y = 11$ and $x - y = 4$, we solve the system of equations obtaining $x = 7.5$ and $y = 3.5$. But x and y are not integers so this solution is inadmissible.

Case 3: $(x + y)(x - y) = 11(9)$
If $x + y = 11$ and $x - y = 9$, we solve the system of equations obtaining $x = 10$ and $y = 1$. But x must be an integer from 1 to 9 so this solution is inadmissible.

Therefore, the only solution is a hypotenuse of length 65.