Problem of the Week
Problem E and Solution
The Hypotenuse is Aligned

Problem

\( \triangle OAB \) is an isosceles right-angled triangle with
- vertex \( O \) located at the origin; and
- vertices \( A \) and \( B \) located on the line \( 2x + 3y - 13 = 0 \) such that \( \angle AOB = 90^\circ \) and \( OA = OB \).

Determine the area of \( \triangle OAB \).

Solution

Solution 1

Let \( B \) have coordinates \((p, q)\). Then the slope of \( OB = \frac{q}{p} \). Since \( \angle AOB = 90^\circ \), then \( OB \perp OA \) and the slope of \( OA \) is the negative reciprocal of the slope of \( OB \). Therefore, the slope of \( OA = \frac{-p}{q} \). Since the triangle is isosceles with \( OA = OB \), it follows that the coordinates of \( A \) are \((-q, p)\). (We can verify this by finding the length of \( OA \) and the length of \( OB \) and showing that both lengths are equal to \( \sqrt{p^2 + q^2} \).)

Since \( B(p, q) \) lies on the line \( 2x + 3y - 13 = 0 \), it satisfies the equation of the line. Therefore, \( 2p + 3q - 13 = 0 \) \( (1) \).

Since \( A(-q, p) \) lies on the line \( 2x + 3y - 13 = 0 \), it satisfies the equation of the line. Therefore, \(-2q + 3p - 13 = 0\), or \(3p - 2q - 13 = 0\) \( (2) \).

Since we have two equations and two unknowns, we can use elimination to solve for \( p \) and \( q \).

\[
(1) \times 2 : \quad 4p + 6q - 26 = 0 \\
(2) \times 3 : \quad 9p - 6q - 39 = 0
\]

Adding, we obtain :
\[
13p - 65 = 0 \\
p = 5
\]

Substituting in (1) :
\[
10 + 3q - 13 = 0 \\
3q = 3 \\
q = 1
\]

Therefore, the point \( B \) is \((5, 1)\) and the length of \( OB \) is \( \sqrt{5^2 + 1^2} = \sqrt{26} \). Since \( OA = OB \), \( OA = \sqrt{26} \).

\( \triangle AOB \) is a right-angled triangle, so we can use \( OB \) as the base and \( OA \) as the height in the formula for the area of a triangle. Therefore, the area of \( \triangle AOB \) is
\[
\frac{OA \times OB}{2} = \frac{\sqrt{26} \sqrt{26}}{2} = 13.
\]

Therefore, the area of \( \triangle AOB \) is 13 units\(^2\).
Solution 2

By rearranging the given equation for the line, we obtain \( y = \frac{-2x+13}{3} \). Since the points \( A \) and \( B \) are on the line, their coordinates satisfy the equation of the line. If \( A \) has \( x \)-coordinate \( a \), then \( A \) has coordinates \( \left( a, \frac{-2a+13}{3} \right) \). If \( B \) has \( x \)-coordinate \( b \), then \( B \) has coordinates \( \left( b, \frac{-2b+13}{3} \right) \). Since \( \triangle OAB \) is isosceles, we know that \( OA = OB \). Then

\[
OA^2 = OB^2
\]

\[
a^2 + \left( \frac{-2a+13}{3} \right)^2 = b^2 + \left( \frac{-2b+13}{3} \right)^2
\]

Multiplying by 9:

\[
9a^2 + 4a^2 - 52a + 169 = 9b^2 + 4b^2 - 52b + 169
\]

Simplifying:

\[
13a^2 - 52a + 169 = 13b^2 - 52b + 169
\]

Rearranging:

\[
13a^2 - 13b^2 - 52a + 52b = 0
\]

Dividing by 13:

\[
a^2 - b^2 - 4a + 4b = 0
\]

Factoring pairs:

\[(a + b)(a - b) - 4(a - b) = 0\]

Common factoring:

\[(a - b)(a + b - 4) = 0\]

Solving, \( a = b \) or \( a = 4 - b \). Since \( A \) and \( B \) are distinct points, \( a \neq b \). Therefore, \( a = 4 - b \).

We can rewrite \( A \left( a, \frac{-2a+13}{3} \right) \) as \( A \left( 4 - b, \frac{-2(4-b)+13}{3} \right) \) which simplifies to \( A \left( 4 - b, \frac{2b+5}{3} \right) \).

Since \( \triangle OAB \) is a right-angled triangle, we can use the Pythagorean Theorem, and \( AB^2 = OA^2 + OB^2 \) follows. But \( OA = OB \), so this can be written \( AB^2 = 2OB^2 \).

\[
AB^2 = 2OB^2
\]

\[
(b - (4 - b))^2 + \left( \frac{-2b+13}{3} - \frac{2b+5}{3} \right)^2 = 2 \left[ b^2 + \left( \frac{-2b+13}{3} \right)^2 \right]
\]

\[
(2b - 4)^2 + \left( \frac{-4b+8}{3} \right)^2 = 2 \left[ b^2 + \frac{4b^2 - 52b + 169}{9} \right]
\]

\[
4b^2 - 16b + 16 + \frac{16b^2 - 64b + 64}{9} = 2b^2 + \frac{8b^2 - 104b + 338}{9}
\]

Multiplying by 9:

\[
36b^2 - 144b + 144 + 16b^2 - 64b + 64 = 18b^2 + 8b^2 - 104b + 338
\]

Simplifying:

\[
52b^2 - 208b + 208 = 26b^2 - 104b + 338
\]

Rearranging:

\[
26b^2 - 104b - 130 = 0
\]

Dividing by 26:

\[
b^2 - 4b - 5 = 0
\]

Factoring:

\[(b - 5)(b + 1) = 0\]

It follows that \( b = 5 \) or \( b = -1 \). When \( b = 5 \), the point \( A \) is \((-1, 5)\) and the point \( B \) is \((5, 1)\).

When \( b = -1 \), the point \( A \) is \((5, 1)\) and the point \( B \) is \((-1, 5)\). There are only two points. The area calculations shown in Solution 1 follow from here.

Therefore, the area of \( \triangle OAB \) is 13 units².