$10^{2021} - 2021$

Problem of the Week
Problem D and Solution
An Exponential Year

Problem

Determine the sum of the digits in the difference when $10^{2021} - 2021$ is evaluated.

Solution

Solution 1

When the number 10^{2021} is written out, there is a one followed by 2021 zeroes, for a total of 2022 digits. Let's look at what happens in our effort to subtract.

$$\begin{array}{r} 1\ 0\ 0\ 0\ 0\ \cdots\ 0\ 0\ 0\ 0\ 0\ 0 \\ -\ 2\ 0\ 2\ 1 \\ \hline \end{array}$$

Using the standard subtraction algorithm, we start with the rightmost digits. In this case we need to borrow. But the borrowing creates a chain reaction. The result after the borrowing is complete is shown below.

$$\begin{array}{r} \not{1}\ \overset{9}{\not{0}}\ \overset{9}{\not{0}}\ \overset{9}{\not{0}}\ \overset{9}{\not{0}}\ \cdots\ \overset{9}{\not{0}}\ \overset{9}{\not{0}}\ \overset{9}{\not{0}}\ \overset{9}{\not{0}}\ \overset{9}{\not{0}}\ \overset{1}{0} \\ -\ 2\ 0\ 2\ 1 \\ \hline 9\ 9\ 9\ 9\ \cdots\ 9\ 9\ 7\ 9\ 7\ 9 \end{array}$$

The four rightmost digits in the difference are 7, 9, 7, and 9. To the left of these digits every digit is a 9. But how many nines are there? The difference has one less digit than 10^{2021}, so has 2021 digits. We have accounted for the four rightmost digits. So to the left of 7979 there are $2021 - 4 = 2017$ nines.

Therefore, the digit sum is

$$2017 \times 9 + (7 + 9 + 7 + 9) = 18\,153 + 32 = 18\,185.$$

Solution 2

The expression $10^{2021} - 2021$ has the same value as $(10^{2021} - 1) - (2021 - 1)$.

As mentioned in Solution 1, when 10^{2021} is written out, there is a one followed by 2021 zeroes, for a total of 2022 digits. The number $(10^{2021} - 1)$ is one less than 10^{2021} and therefore is the positive whole number made up of exactly 2021 nines. When 1 is subtracted from 2021, the difference is 2020. The following is the equivalent subtraction question:

$$\begin{array}{r} 9\ 9\ 9\ 9\ \cdots 9\ 9\ 9\ 9\ 9\ 9 \\ -\ 2\ 0\ 2\ 0 \\ \hline 9\ 9\ 9\ 9\ \cdots 9\ 9\ 7\ 9\ 7\ 9 \end{array}$$

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