Problem of the Week
Problem D and Solution
An Exponential Year

Problem
Determine the sum of the digits in the difference when $10^{2021} - 2021$ is evaluated.

Solution
Solution 1

When the number $10^{2021}$ is written out, there is a one followed by 2021 zeroes, for a total of 2022 digits. Let’s look at what happens in our effort to subtract.

\[
\begin{array}{c}
1 \ 0 \ 0 \ 0 \ 0 \ \cdots \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \\
\end{array}
\]

\[
\begin{array}{c}
- \ 2 \ 0 \ 2 \ 1 \\
\end{array}
\]

Using the standard subtraction algorithm, we start with the rightmost digits. In this case we need to borrow. But the borrowing creates a chain reaction. The result after the borrowing is complete is shown below.

\[
\begin{array}{c}
1 \ 0 \ 9 \ 0 \ 9 \ 9 \ 9 \ \cdots \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \\
\end{array}
\]

\[
\begin{array}{c}
- \ 2 \ 0 \ 2 \ 1 \\
\end{array}
\]

\[
\begin{array}{c}
9 \ 9 \ 9 \ 9 \ \cdots \ 9 \ 9 \ 7 \ 9 \ 7 \ 9 \\
\end{array}
\]

The four rightmost digits in the difference are 7, 9, 7, and 9. To the left of these digits every digit is a 9. But how many nines are there? The difference has one less digit than $10^{2021}$, so has 2021 digits. We have accounted for the four rightmost digits. So to the left of 7979 there are $2021 - 4 = 2017$ nines.

Therefore, the digit sum is

\[
2017 \times 9 + (7 + 9 + 7 + 9) = 18153 + 32 = 18185.
\]
Solution 2

The expression $10^{2021} - 2021$ has the same value as $(10^{2021} - 1) - (2021 - 1)$.

As mentioned in Solution 1, when $10^{2021}$ is written out, there is a one followed by 2021 zeroes, for a total of 2022 digits. The number $(10^{2021} - 1)$ is one less than $10^{2021}$ and therefore is the positive whole number made up of exactly 2021 nines. When 1 is subtracted from 2021, the difference is 2020. The following is the equivalent subtraction question:

\[
\begin{array}{c}
9999 \cdots 9999999 \\
- 2020
\end{array}
\]

\[
\begin{array}{c}
9999 \cdots 997979
\end{array}
\]

The four rightmost digits in the difference are 7, 9, 7 and 9. To the left of these digits every digit is a 9. But how many nines are there? The difference has one less digit than $10^{2021}$, so has 2021 digits. We have accounted for the four rightmost digits. So to the left of 7979 there are $2021 - 4 = 2017$ nines.

Therefore, the digit sum is

$2017 \times 9 + (7 + 9 + 7 + 9) = 18153 + 32 = 18185$. 