Problem of the Week
Problem C and Solution
Locate the Fourth Vertex

Problem
Quadrilateral $BDFH$ is constructed so that each vertex is on a different side of square $ACEG$. Vertex $B$ is on side $AC$ so that $AB = 4$ cm and $BC = 6$ cm. Vertex $F$ is on $EG$ so that $EF = 3$ cm and $FG = 7$ cm. Vertex $H$ is on $GA$ so that $GH = 4$ cm and $HA = 6$ cm. The area of quadrilateral $BDFH$ is $47$ cm$^2$.

The fourth vertex of quadrilateral $BDFH$, labelled $D$, is located on side $CE$ so that the lengths of $CD$ and $DE$ are both positive integers.

Determine the lengths of $CD$ and $DE$.

Solution
Solution 1

Since $ACEG$ is a square and the length of $AG = AH + HG = 6 + 4 = 10$ cm, then the length of each side of the square is 10 cm and the area is $10 \times 10 = 100$ cm$^2$.

We can determine the area of triangles $BAH$ and $FGH$ using the formula $\text{area} = \frac{\text{base} \times \text{height}}{2}$.

In $\triangle BAH$, since $BA$ is perpendicular to $AH$, we can use $BA$ as the height and $AH$ as the base. The area of $\triangle BAH$ is $\frac{6 \times 4}{2} = 12$ cm$^2$.

In $\triangle FGH$, since $FG$ is perpendicular to $GH$, we can use $FG$ as the height and $GH$ as the base. The area of $\triangle FGH$ is $\frac{4 \times 7}{2} = 14$ cm$^2$.

The area of $\triangle BCD$ plus the area of $\triangle FED$ must be the total area minus the three known areas. That is, $\text{Area } \triangle BCD + \text{Area } \triangle FED = 100 - 12 - 47 - 14 = 27$ cm$^2$.

$CD$ and $DE$ are both positive integers and $CD + DE = 10$. We will systematically check all possible values for $CD$ and $DE$ to determine the values which produce the correct area.

<table>
<thead>
<tr>
<th>$CD$</th>
<th>$DE$</th>
<th>Area $\triangle BCD$</th>
<th>Area $\triangle FED$</th>
<th>Area $\triangle BCD + \triangle FED$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9</td>
<td>$1 \times 6 \div 2 = 3$</td>
<td>$9 \times 3 \div 2 = 13.5$</td>
<td>$3 + 13.5 = 16.5 \neq 27$</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>$2 \times 6 \div 2 = 6$</td>
<td>$8 \times 3 \div 2 = 12$</td>
<td>$6 + 12 = 18 \neq 27$</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>$3 \times 6 \div 2 = 9$</td>
<td>$7 \times 3 \div 2 = 10.5$</td>
<td>$9 + 10.5 = 19.5 \neq 27$</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>$4 \times 6 \div 2 = 12$</td>
<td>$6 \times 3 \div 2 = 9$</td>
<td>$12 + 9 = 21 \neq 27$</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>$5 \times 6 \div 2 = 15$</td>
<td>$5 \times 3 \div 2 = 7.5$</td>
<td>$15 + 7.5 = 22.5 \neq 27$</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>$6 \times 6 \div 2 = 18$</td>
<td>$4 \times 3 \div 2 = 6$</td>
<td>$18 + 6 = 24 \neq 27$</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>$7 \times 6 \div 2 = 21$</td>
<td>$3 \times 3 \div 2 = 4.5$</td>
<td>$21 + 4.5 = 25.5 \neq 27$</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>$8 \times 6 \div 2 = 24$</td>
<td>$2 \times 3 \div 2 = 3$</td>
<td>$24 + 3 = 27$</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>$9 \times 6 \div 2 = 27$</td>
<td>$1 \times 3 \div 2 = 1.5$</td>
<td>$27 + 1.5 = 28.5 \neq 27$</td>
</tr>
</tbody>
</table>

Therefore, when $CD = 8$ cm and $DE = 2$ cm, the area of quadrilateral $BDFH$ is $47$ cm$^2$.

The second solution is more algebraic and will produce a solution for any lengths of $CD$ and $DE$ between 0 and 10 cm.
Solution 2

This solution begins the same as Solution 1. Algebra is introduced to complete the solution.

Since $ACEG$ is a square and the length of $AG = AH + HG = 6 + 4 = 10$ cm, then the length of each side of the square is 10 cm and the area is $10 \times 10 = 100$ cm$^2$.

We can determine the area of the triangles $BAH$ and $FGH$ using the formula $\frac{\text{base} \times \text{height}}{2}$.

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The area of $\triangle BCD$ plus the area of $\triangle FED$ must be the total area minus the three known areas. That is, $\text{Area } \triangle BCD + \text{Area } \triangle FED = 100 - 12 - 47 - 14 = 27$ cm$^2$.

Let the length of $CD$ be $n$ cm. Then the length of $DE$ is $(10 - n)$ cm.

The area of $\triangle BCD$ is $\frac{BC \times CD}{2} = \frac{6 \times n}{2} = 3n$.

The area of $\triangle FED$ is $\frac{FE \times DE}{2} = \frac{3 \times (10 - n)}{2} = \frac{10 - n + 10 - n + 10 - n}{2} = \frac{30 - 3n}{2}$.

Therefore,

\[
\text{Area } \triangle BCD + \text{Area } \triangle FED = 27 \\
3n + \frac{30 - 3n}{2} = 27 \\
\]

Multiplying both sides by 2:

\[
6n + 30 - 3n = 54 \\
3n + 30 = 54 \\
3n = 24 \\
n = 8
\]

Therefore, the length of $CD$ is 8 cm and the length of $DE$ is 2 cm.

The algebra presented in Solution 2 may not be familiar to all students at this level.