Problem of the Week
Problem C and Solution
Order Up!

Problem
The letters \( w, x, y, \) and \( z \) each represent a different positive integer greater than 3. If we know that

\[
\frac{1}{w-3} = \frac{1}{x+1} = \frac{1}{y+2} = \frac{1}{z-2}
\]

then write \( w, x, y, \) and \( z \) in order from the letter that represents the smallest integer to the letter that represents the largest integer.

Solution
Solution 1:
Since the fractions are all equal and they all have a numerator of 1, that means that their denominators must all be equal. So \( w - 3 = x + 1 = y + 2 = z - 2 \).

Now let’s suppose that \( w = 10 \). Then \( w - 3 = 10 - 3 = 7 \).

So \( 7 = x + 1 = y + 2 = z - 2 \). We can make the following conclusions.

- Since \( 7 = x + 1 \), that means \( x = 7 - 1 = 6 \).
- Since \( 7 = y + 2 \), that means \( y = 7 - 2 = 5 \).
- Since \( 7 = z - 2 \), that means \( z = 7 + 2 = 9 \).

So when \( w = 10 \), we have \( x = 6 \), \( y = 5 \), and \( z = 9 \). We can see that \( x \) is four less than \( w \), \( y \) is five less than \( w \), and \( z \) is one less than \( w \). So when we write these in order from smallest to largest, we get \( y, x, z, w \).

Solution 2:
As with Solution 1, we notice that since the fractions are all equal and they all have a numerator of 1, that means that their denominators must all be equal.

So \( w - 3 = x + 1 = y + 2 = z - 2 \). Let’s add 3 to each expression.

\[
\begin{align*}
w - 3 &= x + 1 &= y + 2 &= z - 2 \\
\downarrow +3 & & \downarrow +3 & & \downarrow +3 & & \downarrow +3 \\
w &= x + 4 &= y + 5 &= z + 1
\end{align*}
\]

From this we can make the following conclusions.

- Since \( w = z + 1 \), that means \( w \) is 1 more than \( z \), so \( w > z \).
- Since \( z + 1 = x + 4 \), that means \( z \) is 3 more than \( x \), so \( z > x \).
- Since \( x + 4 = y + 5 \), that means \( x \) is 1 more than \( y \), so \( x > y \).

So when we write these in order from smallest to largest, we get \( y, x, z, w \).