Problem of the Week
Problem E and Solution
Two Circles and a Triangle

Problem
Two circles, with centres $O$ and $B$ and each with a radius of 2, are tangent to each other. A straight line is drawn through $O$ and $B$ meeting the circles at $Q$ and $R$. Two other sides of $\triangle PQR$ are drawn such that side $PR$ is tangent to the circle with centre $B$ at $A$ and side $PQ$ is tangent to the circle with centre $B$ at $Q$. Determine the length of $PQ$.

Solution
Let $T$ be the point of tangency of the two circles. Then $T$ lies on $OB$, the line segment joining the two centres. Also, $QB = BT = TO = OR = 2$. Since $PR$ is tangent to the circle with centre $B$ at $A$, $AB \perp PR$. Since $PQ$ is tangent to the circle with centre $B$ at $Q$, $PQ \perp QB$. Since $AB$ and $QB$ are radii of the circle with centre $B$, $QB = AB = 2$. This information has been added to the diagram.

$\triangle BAR$ is right angled at $A$ since $AB \perp AR$. Therefore, $AR^2 = BR^2 - AB^2 = 6^2 - 2^2 = 36 - 4 = 32$. Since $AR > 0$, $AR = \sqrt{32} = 4\sqrt{2}$.

At this point there are many ways to find the length of $PQ$. We will look at three solutions.

Solution 1
An interesting (and possibly different) way to find $PQ$ is to use basic trigonometry in the right triangles. In $\triangle BAR$, $\tan(R) = \frac{AB}{AR}$ and in $\triangle PQR$, $\tan(R) = \frac{PQ}{QR}$.

\[
\therefore \frac{AB}{AR} = \frac{PQ}{QR}
\]

\[
\frac{2}{4\sqrt{2}} = \frac{PQ}{8}
\]

\[
PQ = \frac{16}{4\sqrt{2}} = \frac{4}{\sqrt{2}} = \frac{4\times\sqrt{2}}{2} = 2\sqrt{2}
\]

Therefore, the length of $PQ$ is $2\sqrt{2}$. 
Solution 2

Construct a line from $B$ to $P$. We will show that $PQ = PA$.

Both $\triangle PQB$ and $\triangle PAB$ are right triangles. Using the Pythagorean Theorem,

$PB^2 = PQ^2 + QB^2$ \hspace{1cm} and \hspace{1cm} $PB^2 = PA^2 + AB^2$

$\therefore PQ^2 + QB^2 = PA^2 + AB^2$

$PQ^2 + (2)^2 = PA^2 + (2)^2$

It follows that $PQ^2 = PA^2$

And $PQ = PA$, since $PQ > 0$ and $PA > 0$

Let $PQ = PA = x$. The information is added to the diagram.

Since $PQ \perp QB$, $\triangle PQR$ is a right triangle with $PQ = x$,
$QR = QB + BT + TO + OR = 8$ and $PR = PA + AR = 4\sqrt{2} + x$.

Using the Pythagorean Theorem in $\triangle PQR$,

$PR^2 = QR^2 + PQ^2$

$(4\sqrt{2} + x)^2 = 8^2 + x^2$

$32 + (8\sqrt{2})x + x^2 = 64 + x^2$

$(8\sqrt{2})x = 32$

$x = \frac{32}{8\sqrt{2}}$

$x = \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$x = 2\sqrt{2}$

Therefore, the length of $PQ$ is $2\sqrt{2}$. 
Solution 3

We will first show that $\triangle BAR$ is similar to $\triangle PQR$.

In $\triangle BAR$ and $\triangle PQR$

\[ \angle ARB = \angle QRP \quad \text{(common angle)} \]
\[ \angle BAR = \angle PQR \quad \text{(both are right angles)} \]

$\therefore \triangle BAR \sim \triangle PQR$ (AA $\triangle$ similarity)

and $\frac{AR}{QR} = \frac{AB}{PQ}$ follows.

\[ \frac{4\sqrt{2}}{8} = \frac{2}{PQ} \]

\[ (4\sqrt{2})PQ = 16 \]

\[ PQ = \frac{16}{4\sqrt{2}} \]

\[ = \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \]

\[ = 2\sqrt{2} \]

Therefore, the length of $PQ$ is $2\sqrt{2}$. 