Problem of the Week
Problem E and Solution
What’s Outside?

Problem
In the diagram above, \( \triangle PQR \) is an equilateral triangle with sides of length 2 cm. Arc \( PQ \) is an arc of a circle with centre \( R \) and radius \( RQ \). Arc \( PR \) and arc \( RQ \) are similarly drawn with centres \( Q \) and \( P \), respectively. Determine the total area of the shaded regions in the diagram.

Solution

First, determine the area of \( \triangle PQR \). Since the triangle is equilateral, each of the angles in the triangle are 60\(^\circ\). Construct altitude \( PT \), shown to the right. Then \( \triangle PTR \) is a 30\(^\circ\)-60\(^\circ\)-90\(^\circ\) triangle with sides in the ratio 1 : \( \sqrt{3} \) : 2.

Since \( QR = 2 \) cm, it follows that \( TR = 1 \) cm and \( PT = \sqrt{3} \) cm. Therefore,
\[
\text{area} \ \triangle PQR = (QR)(PT) \div 2 \]
\[
= (2)(\sqrt{3}) \div 2 \]
\[
= \sqrt{3} \text{ cm}^2
\]

The diagram consists of three overlapping circle sectors, one with centre \( P \), one with centre \( Q \), and one with centre \( R \). Each circle sector has the same radius, 2 cm, and a 60\(^\circ\) central angle. Therefore each sector has the same area, 60 \( \div 360 \) or one-sixth the area of a circle of radius 2 cm. That is,
\[
\text{area of each sector} = \frac{1}{6} \pi r^2 = \frac{1}{6} \pi (2)^2 = \frac{2}{3} \pi \text{ cm}^2
\]

The shaded part of each circle sector is equal to the area of the sector minus the area of \( \triangle PQR \). Since there are three congruent shaded areas,
\[
\text{total shaded area} = 3 \left( \text{area of any whole circle sector} - \text{area of } \triangle PQR \right)
\]
\[
= 3 \left( \frac{2}{3} \pi - \sqrt{3} \right)
\]
\[
= (2\pi - 3\sqrt{3}) \text{ cm}^2
\]

Therefore, the total area of the shaded regions is equal to \((2\pi - 3\sqrt{3}) \text{ cm}^2\).