Problem

A local food bank has created a unique 100-day plan for collecting canned food donations.

**Day 1 Goal:** Collect 50 cans of food.

**Day 2 Goal:** Collect 3 more cans of food than the current day number plus the same number of cans collected on day 1.

**Day 3 Goal:** Collect 3 more cans of food than the current day number plus the same number of cans collected on day 2.

**Day 4 Goal:** Collect 3 more cans of food than the day number plus the same number of cans collected on day 3.

...  

**Day 100 Goal:** Collect 3 more cans of food than the day number plus the same number of cans collected on day 99.

How many cans of food will the food bank collect on the 100th day?

**Extension:** Assuming their target is met each day of the 100-day campaign, how many cans of food will they collect in total?

Solution

First we will introduce function notation to represent the information in the problem.

Let \( n \) represent the day number and \( f(n) \) represent the number of cans collected on day \( n \).

We know that on the first day, 50 cans were collected. So, \( f(1) = 50 \).

On the second day, they collect 3 more cans of food than the current day number plus the same number of cans collected on day 1. So, \( f(2) = 3 + 2 + f(1) = 3 + 2 + 50 = 55 \).

On the third day, they collect 3 more cans of food than the current day number plus the same number of cans collected on day 2. So, \( f(3) = 3 + 3 + f(2) = 3 + 3 + 55 = 61 \).

On the fourth day, they collect 3 more cans of food than the day number plus the same number of cans collected on day 3. So, \( f(4) = 3 + 4 + f(3) = 3 + 4 + 61 = 68 \).

On the \( n \)th day, they collect 3 more cans of food than the current day number plus the same number of cans collected on day \( (n - 1) \). So, \( f(n) = 3 + n + f(n - 1) \).

This sequence of cans collected can be defined recursively as follows:

\[
f(n) = \begin{cases} 
50, & \text{if } n = 1, \\
 f(n-1) + n + 3, & n \geq 2, n \in \mathbb{Z}.
\end{cases}
\]

We want to find \( f(100) \), the 100th term in the sequence, and if we complete the extension, we want \( f(1) + f(2) + f(3) + \cdots + f(100) \). Several approaches follow on the next pages.
Approach 1

One obvious, yet exhausting, approach to determining the value collected on the 100th day would be to determine the values of all the preceding days. This is not really practical and will not be pursued here.

Approach 2

Earlier we noted that \( f(1) = 50, \ f(2) = 55, \ f(3) = 61, \) and \( f(4) = 68. \) From \( f(1) \) to \( f(2) \) the sequence of cans collected increased by 5, from \( f(2) \) to \( f(3) \) the sequence increased by 6, and from \( f(3) \) to \( f(4) \) the sequence increased by 7. We see a pattern and might predict that going from \( f(4) \) to \( f(5) \) the sequence would increase by 8. Calculating \( f(5) \) using the function we get

\[
 f(5) = f(4) + 5 + 3 = 68 + 8 = 76.
\]

Our prediction was correct but does it generalize?

To justify this further, if \( f(p−1) \) and \( f(p) \) are adjacent terms in the sequence of cans collected, then

\[
 f(p) = f(p−1) + p + 3.
\]

Rearranging,

\[
 f(p) − f(p−1) = p + 3.
\]

That is, the difference between consecutive terms will be the term number of the term in the higher position plus 3. So

\[
 f(6) − f(5) = 6 + 3 = 9 \quad \text{and} \quad f(7) − f(6) = 7 + 3 = 10.
\]

Can we use this to get from the first term, \( f(1) \), to the the hundredth term, \( f(100) \)? Going from the first term to the hundredth term we advance 99 terms. The value of the hundredth term would be 50, the value of the first term, plus 99 consecutive integers starting with 5.

That sum is \( 50 + (5 + 6 + 7 + 8 + \cdots + 100 + 101 + 102 + 103) \)

\[
 = 50 + (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + \cdots + 100 + 101 + 102 + 103) − (1 + 2 + 3 + 4).
\]

Using the formula \( \frac{n(n+1)}{2} \) with \( n = 103 \) and \( n = 4 \), this becomes

\[
 50 + \frac{103(104)}{2} − \frac{4(5)}{2} = 50 + 5356 − 10 = 5396.
\]

That is, 5396 cans are collected on the 100th day.
Approach 3

The content in this approach may be unfamiliar to many of you reading it.

From our work so far, we create a table:

<table>
<thead>
<tr>
<th>Day Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>50</td>
<td>55</td>
<td>61</td>
<td>68</td>
<td>76</td>
<td>85</td>
</tr>
</tbody>
</table>

If we treat the day number as the independent variable, say $x$, and the value as the dependent variable, say $y$, we can make some conclusions. First, $\Delta x$ is constant. However, $\Delta y$, the first differences are not constant. Therefore, the function is not linear.

The first differences of the given values are 5, 6, 7, 8, and 9. If we calculate the second differences, each of them is 1. That is, for a constant $\Delta x$, the first differences are not constant but the second differences are constant. This tells us that the function is quadratic, that is, $f(x) = ax^2 + bx + c, x \geq 1, x \in \mathbb{Z}$. The symbol $\mathbb{Z}$ is used for the set of integers.

We know that $f(1) = a(1)^2 + b(1) + c = a + b + c$ and $f(1) = 50$, so

$$a + b + c = 50 \quad (1)$$

We know that $f(2) = a(2)^2 + b(2) + c = 4a + 2b + c$ and $f(2) = 55$, so

$$4a + 2b + c = 55 \quad (2)$$

We know that $f(3) = a(3)^2 + b(3) + c = 9a + 3b + c$ and $f(3) = 61$, so

$$9a + 3b + c = 61 \quad (3)$$

We will eliminate $c$ from the system of equations.

$$(2) - (1) \quad 3a + b = 5 \quad (4)$$

$$(3) - (2) \quad 5a + b = 6 \quad (5)$$

Now, eliminate $b$ from the system of equations.

$$(5) - (4) \quad 2a = 1 \Rightarrow a = \frac{1}{2}$$

Substitute in (4) to find $b$: $3 \left(\frac{1}{2}\right) + b = 5 \Rightarrow b = 5 - \frac{3}{2} = \frac{7}{2}$

Substitute in (1) to find $c$: $\frac{1}{2} + \frac{7}{2} + c = 50 \Rightarrow c = 50 - 4 = 46$

Therefore, the quadratic function is $f(x) = \frac{1}{2}x^2 + \frac{7}{2}x + 46, x \geq 1, x \in \mathbb{Z}$.

To determine the value collected on the 100th day, evaluate $f(x)$ for $x = 100$.

$$f(100) = \frac{1}{2} (100)^2 + \frac{7}{2} (100) + 46 = 5000 + 350 + 46 = 5396$$

That is, 5396 cans are collected on the 100th day.

Using this approach, we can find the amount collected on any day, since we have a general formula for the value collected given the day number. That is, the amount collected on day $n$ is $f(n) = \frac{1}{2} n^2 + \frac{7}{2} n + 46$, where $n \geq 1, n \in \mathbb{Z}$. 
Approach 4

The content in this approach will probably be unfamiliar to most of you reading it.

We are given this following definition:

\[
f(n) = \begin{cases} 
  50, & \text{if } n = 1, \\
  f(n-1) + n + 3, & \text{if } n \geq 2, n \in \mathbb{Z}.
\end{cases}
\]

Rearranging the definition of the function for \( n \geq 2 \) we obtain:

\[
f(n) - f(n-1) = n + 3
\]

Using \( f(n) - f(n-1) = n + 3 \) with different values of \( n \):

When \( n = 100 \), \( f(100) - f(99) = 100 + 3 \)
When \( n = 99 \), \( f(99) - f(98) = 99 + 3 \)
When \( n = 98 \), \( f(98) - f(97) = 98 + 3 \)

\[...\]
When \( n = 4 \), \( f(4) - f(3) = 4 + 3 \)
When \( n = 3 \), \( f(3) - f(2) = 3 + 3 \)
When \( n = 2 \), \( f(2) - f(1) = 2 + 3 \)

If we add all of the terms on the left side of each equal sign, we are left with only \( f(100) - f(1) \) since we have \(-f(99) + f(99), -f(98) + f(98), \ldots, -f(3) + f(3), \) and \(-f(2) + f(2)\).

If we add all of the terms on the right side of each equal sign, we get

\[
2 + 3 + 4 + \cdots + 98 + 99 + 100 + 99(3)
\]

Therefore, \( f(100) - f(1) = 2 + 3 + 4 + \cdots + 98 + 99 + 100 + 99(3) \)

Since \( f(1) = 50 \), \( f(100) - 50 = 5049 + 297 \)
\[
\begin{align*}
  f(100) &= 5049 + 297 + 50 \\
         &= 5396
\end{align*}
\]

That is, 5396 cans are collected on the 100th day.

Note concerning (*) above:

\[
2 + 3 + 4 + \cdots + 98 + 99 + 100
= 1 + 2 + 3 + 4 + \cdots + 98 + 99 + 100 - 1
= \frac{100(101)}{2} - 1
= 5050 - 1
= 5049
\]