Problem of the Week
Problem E and Solution
This is the Year 2

Problem
The positive even integers are arranged in increasing order in a triangle, as shown:

\[
\begin{array}{ccc}
2 \\
4 & 6 \\
8 & 10 & 12 \\
14 & 16 & 18 & 20 \\
22 & 24 & 26 & \cdots
\end{array}
\]

Each row contains one more number than the previous row. One of the rows of the triangle contains the number 2020. Determine the sum of the numbers in the row that contains 2020.

Solution
To begin the solution, we make an observation: the row number in the triangle is the same as the number of numbers in that particular row. For example, row 1 contains 1 number, row 2 contains 2 numbers, and row \( r \) contains \( r \) numbers. If we add up all of the row numbers from row 1 to row \( r \) we get the number of numbers in the table and the largest number in the row is \( 2r \).

For example, if we calculate the sum \( 1 + 2 + 3 + 4 \), we get 10. This number corresponds to the number of numbers in the first four rows of the triangle and the largest number in the fourth row is \( 2 \times 10 = 20 \).

Let \( n \) represent the row number of the row that contains the number 2020. Since \( 2020 = 2 \times 1010 \) we need to find the row where the 1010th number occurs.

Using the formula for the sum of the integers from 1 to \( n \), we want \( \frac{n(n+1)}{2} \geq 1010 \). Multiplying by 2, the equation simplifies to \( n(n + 1) \geq 2020 \).

We will use trial and error to find the possible value of \( n \). Finding the \( \sqrt{2020} \) will get us a good place to start checking values of \( n \). (\( \sqrt{2020} \approx 44.9 \).)

When \( n = 44 \), \( 44 \times 45 = 1980 < 2020 \) and \( \frac{44 \times 45}{2} = 990 \).

When \( n = 45 \), \( 45 \times 46 = 2070 > 2020 \) and \( \frac{45 \times 46}{2} = 1035 \).

These guesses are quite useful. When there are 44 rows in the triangle, the last number in the 44th row is 990 \( \times 2 = 1980 \). So the first number in the 45th row is 1982. When there are 45 rows in the triangle, the last number in the 45th row is 2 \( \times 1035 = 2070 \). We want the sum of the numbers in the 45th row. To do this we will add all of the even integers from 2 to 2020. But this sum includes extra even integers from 2 to 1980. We will use our formula to compute this second sum and subtract it from the first sum to obtain the desired result.
\[
\text{sum} = 1982 + 1984 + 1986 + \cdots + 2066 + 2068 + 2070
\]
\[
= \left( 2 + 4 + 6 + \cdots + 1976 + 1978 + 1980 \right) + 1982 + 1984 + 1986 + \cdots + 2066 + 2068 + 2070
\]
\[
- \left( 2 + 4 + 6 + \cdots + 1976 + 1978 + 1980 \right)
\]
\[
= \left( 2(1 + 2 + 3 + \cdots + 988 + 989 + 990) + 991 + 992 + 993 + \cdots + 1033 + 1034 + 1035 \right)
\]
\[
- \left( 2(1 + 2 + 3 + \cdots + 988 + 989 + 990) \right)
\]
\[
= 2 \times \left( \frac{1035 \times 1036}{2} \right) - 2 \times \left( \frac{990 \times 991}{2} \right)
\]
\[
= 91170
\]

The sum of the numbers in the row containing 2020 is 91170.