Problem of the Week
Problem D and Solution
One Step at a Time

Problem

A particular sequence begins with first term 2. To obtain the next term in the sequence from the term immediately before it, multiply the preceding term by 3, then add 2 to your result and finally divide by this new result by 3. Repeat this set of steps with each new term to generate more terms in the sequence.

Following these steps with first term 2, the value of the second term is $\frac{8}{3}$ and the term number is 2. Following the steps with the second term, you will obtain term 3 whose value is $\frac{10}{3}$. Determine the value of the 1000$^{\text{th}}$ term.

Term Number	1	2	3	$\cdots$ $\cdots$	1000
Value	2	$\frac{8}{3}$	$\frac{10}{3}$	$\cdots$ $\cdots$	????

Solution

Solution 1

In order to find the 1000$^{\text{th}}$ term, when using the definition, we would need the 999$^{\text{th}}$ term. To find the 999$^{\text{th}}$ term, we would need the 998$^{\text{th}}$ term. And to find the 998$^{\text{th}}$ term, we would need the 997$^{\text{th}}$ term; and so on. Using this method is not practical but it would lead to the 1000$^{\text{th}}$ term 668.

However, we can sometimes develop a rule that allows us to determine the value of a term that only depends on knowing the term number (its position in the sequence). We will do this in the following solutions.

Solution 2

To start, we will generate a few more terms.

For term 4: $\frac{10}{3} \Rightarrow \boxed{\text{Multiply by 3}} \Rightarrow 10 \Rightarrow \boxed{\text{Add 2}} \Rightarrow 12 \Rightarrow \boxed{\text{Divide by 3}} \Rightarrow \frac{12}{3} = 4$

For term 5: $4 \Rightarrow \boxed{\text{Multiply by 3}} \Rightarrow 12 \Rightarrow \boxed{\text{Add 2}} \Rightarrow 14 \Rightarrow \boxed{\text{Divide by 3}} \Rightarrow \frac{14}{3}$

For term 6: $\frac{14}{3} \Rightarrow \boxed{\text{Multiply by 3}} \Rightarrow 14 \Rightarrow \boxed{\text{Add 2}} \Rightarrow 16 \Rightarrow \boxed{\text{Divide by 3}} \Rightarrow \frac{16}{3}$

For term 7: $\frac{16}{3} \Rightarrow \boxed{\text{Multiply by 3}} \Rightarrow 16 \Rightarrow \boxed{\text{Add 2}} \Rightarrow 18 \Rightarrow \boxed{\text{Divide by 3}} \Rightarrow \frac{18}{3} = 6$

The sequence begins $2, \frac{8}{3}, \frac{10}{3}, 4, \frac{14}{3}, \frac{16}{3}, 6, \cdots$.

Notice that the difference between consecutive terms appears to always be $\frac{2}{3}$. We will present a justification as to why this is true here.

If p is any term in the sequence and q is the term immediately following it, then we get q by multiplying p by 3, adding 2 to the product, and then dividing the result by 3. That is, $q = \frac{3p+2}{3} = \frac{3p}{3} + \frac{2}{3} = p + \frac{2}{3}$. Then $q = p + \frac{2}{3}$ or $q - p = \frac{2}{3}$. That is, the difference between consecutive terms is $\frac{2}{3}$.

Term 1 and term 1000 are 999 terms apart. So, to get from term 1 to term 1000, we would add 999 multiples of $\frac{2}{3}$ to the first term. Therefore, the 1000$^{\text{th}}$ term is $2 + 999 \times \frac{2}{3} = 2 + 666 = 668$.

Solution 3

We will start this solution with the sequence of terms generated in solution 2.

The sequence is $2, \frac{8}{3}, \frac{10}{3}, 4, \frac{14}{3}, \frac{16}{3}, 6, \cdots$.

Notice that the difference between consecutive terms is $\frac{2}{3}$. A justification of this was presented in Solution 2.

Also note that terms 1, 4 and 7 are the positive integers 2, 4 and 6. The value of any term in these positions is 2 greater than the value of the term 3 positions before it. In fact, the value of any term in position $(1 + 3n)$ for non-negative integer values of n will be $(2 + 2n)$.

We are asked for term 1000. The n that satisfies $1 + 3n = 1000$ is $n = 333$.

When $n = 333$, $2 + 2n = 2 + 2(333) = 668$. Therefore, the 1000^{th} term is 668.

Solution 4

Using the terms generated in solution 2, we can create the following chart.

Term Number	1	2	3	4	5	6	7
Value	$2 = \frac{6}{3}$	$\frac{8}{3}$	$\frac{10}{3}$	$4 = \frac{12}{3}$	$\frac{14}{3}$	$\frac{16}{3}$	$6 = \frac{18}{3}$

Notice that the first, fourth and seventh terms have been re-written as a fraction with denominator 3.

Looking at each term, observe that the denominators are all 3 and the numerators increase by 2. To get to any term in the sequence, we could add 2 times the term number to the numerator 4 and keep the denominator as 3. That is, if n is the term number, then term n, usually written t_n, would be $\frac{4+2n}{3}$.

For the 1000^{th} term, we substitute $n = 1000$ into the expression for the general term. It follows that the 1000^{th} term is $t_{1000} = \frac{4+2(1000)}{3} = \frac{2004}{3} = 668$.

Solution 5

Let x represent the term number such that x a a positive integer.

Let y represent the value of the term in position x.

x	1	2	3	4	5	6	7
y	2	$\frac{8}{3}$	$\frac{10}{3}$	4	$\frac{14}{3}$	$\frac{16}{3}$	6

As the value of the independent variable x increases by 1, the value of the dependent variable y increases by $\frac{2}{3}$. This data can therefore be modelled with a linear function of the form $y = mx + b$.

Using $(1, 2)$ and $(4, 4)$, we can determine that $m = \dfrac{4-2}{4-1} = \dfrac{2}{3}$.

Substituting $x = 1,\ y = 2,$ and $m = \dfrac{2}{3}$ into

$$\begin{aligned} y &= mx + b \\ \text{we obtain:} \qquad 2 &= \frac{2}{3}(1) + b \\ \text{multiplying by 3:} \qquad 6 &= 2 + 3b \\ \text{solving for } b\text{:} \qquad 4 &= 3b \\ \frac{4}{3} &= b \end{aligned}$$

So, for positive integer values of x, we can use $y = \dfrac{2}{3}x + \dfrac{4}{3}$ to generate terms in the sequence.

For the value of the 1000^{th} term, substitute $x = 1000$ into $y = \dfrac{2}{3}x + \dfrac{4}{3}$.

Then $y = \dfrac{2}{3}(1000) + \dfrac{4}{3} = \dfrac{2000}{3} + \dfrac{4}{3} = \dfrac{2004}{3} = 668$.

Therefore the 1000^{th} term in the sequence is 668.

Note, you may see this general term written $t_n = \dfrac{2n+4}{3}$ where n, the term number, is a positive integer and t_n represents the value of the term in that position.