Problem of the Week
Problem D and Solution

Initial This

Problem
In this two-player game, the vertices of a regular hexagon are each covered by a circle. On a turn, a player may either initial one circle or initial two adjacent circles. (The two adjacent circles must be directly connected by an outside edge of the hexagon.) Players alternate turns. The player initialing the last circle or last pair of adjacent circles is the winner. Two players, Cameron and Dale, play the game with Cameron going first. One of the players, Cameron or Dale, can always win the game. Describe which player can always win and the winning strategy for that player.

Solution
There are two types of possible first moves for Cameron.

(a) Cameron could initial exactly one blank circle.
(b) Cameron could initial two adjacent blank circles.

Let us look at (a) first. We can rotate the hexagon without changing the game, so we will assume that Cameron initials the circle on the left with a ‘C’. This is shown on the diagram to the right.

In this case, Dale should initial the circle on the opposite side of the hexagon with a ‘D’ as shown in the second diagram to the right.

On his second turn, Cameron may initial two adjacent circles, either at the top or bottom. This is shown on the diagram to the right.

Dale should then initial the two remaining adjacent circles on the opposite side to win the game. This is shown on the diagram to the right.
Instead, on his second turn, Cameron may initial only one circle. A possible selection is shown to the right.

Dale should then initial one of the two adjacent unmarked circles on the other side of the hexagon. A possible selection is shown to the right.

Now there are two non-adjacent unmarked circles left. On his third turn, Cameron must initial one of the unmarked circles.

Dale then wins the game by initialing the last unmarked circle.

What happens in situation (b) where Cameron initials two adjacent unmarked circles? Again, we can rotate the hexagon without changing the game, so we will assume that Cameron initials the two unmarked circles at the top.

In this case, Dale should mark the two adjacent unmarked circles on the bottom.

As in the first situation, there are two unmarked circles left and they are not adjacent, so Cameron must initial exactly one of the unmarked circles.

Dale then initials the last unmarked circle to win the game.

We have considered all of the possible cases and have shown that Dale has a winning strategy. Dale should “copy” Cameron by initialing the same number of unmarked circles as Cameron but on the opposite side of the hexagon. If Dale follows this strategy, Dale will always win.

**For Further Thought**

Without changing the rules of the game, how would the strategy change if instead of a hexagon with 6 circles, there were a heptagon with 7 circles?