Problem of the Week
Problem D and Solution
Build Around

Problem
Quadrilateral $ABCD$ is built around a circle with centre $O$ and radius 15 cm so that each side of $ABCD$ is tangent to the circle, sides $AD$ and $BC$ are parallel, and side lengths $AB$ and $DC$ are equal. $ABCD$ is called an isosceles trapezoid. If the area of $ABCD$ is $1000\text{ cm}^2$, determine the lengths of the two equal sides, $AB$ and $DC$.

Solution
Solution 1
Let $z$ represent the length of side $AD$ and $y$ represent the length of side $BC$.

Draw a line segment through $O$ perpendicular to both $AD$ and $BC$.

Using the second result given after the problem statement, we know that this perpendicular meets $AD$ at the point of tangency $S$ and $BC$ at the point of tangency $Q$.

Both $OS$ and $OQ$ are radii of the circle. Since $SQ = OS + OQ$, it follows that $SQ$ is a diameter of the circle and has length 30 cm. Since $SQ$ is perpendicular to the two parallel sides of the trapezoid, we can use $SQ$ as the height of the trapezoid.

Using the formula for the area of a trapezoid, we obtain

$$\text{Area} = \frac{SQ \times (AD + BC)}{2}$$

$$1000 = \frac{30 \times (z + y)}{2}$$

$$1000 = 15(z + y)$$

$$1000 = \frac{200}{3} = z + y \quad (1)$$
Let the length of the two equal sides, $AB$ and $CD$, be $x$.

Join the centre $O$ to each of the vertices of $ABCD$, creating four triangles, $\triangle AOB$, $\triangle BOC$, $\triangle COD$, and $\triangle DOA$.

Connect the centre $O$ to each of the points of tangency, $P$, $Q$, $R$, and $S$, on $AB$, $BC$, $CD$, and $DA$, respectively.

Each of these segments is a radius so $OP = OQ = OR = OS = 15$. From the first result given after the problem statement, we know that each of these segments is perpendicular to the tangent at the point of tangency so each of these radii can be a height of their respective triangles.

We can now find the area of the trapezoid a second way by summing the areas of the four triangles:

$\text{Area } \triangle AOB = OP \times AB \div 2 = \frac{15x}{2}$, \hspace{1cm} $\text{Area } \triangle BOC = OQ \times BC \div 2 = \frac{15y}{2}$

$\text{Area } \triangle COD = OR \times CD \div 2 = \frac{15x}{2}$ \hspace{1cm} $\text{Area } \triangle DOA = OS \times AD \div 2 = \frac{15z}{2}$

\[
\text{Area } \triangle AOB + \text{Area } \triangle BOC + \text{Area } \triangle COD + \text{Area } \triangle DOA = 1000 \\
\frac{15x}{2} + \frac{15y}{2} + \frac{15x}{2} + \frac{15z}{2} = 1000 \\
15x + \frac{15y}{2} + \frac{15z}{2} = 1000 \\
15x + \frac{15y + 15z}{2} = 1000 \\
15x + \frac{15(y + z)}{2} = 1000 \\
\text{But } y + z = \frac{200}{3} \text{ from (1) above so } 15x + \frac{15\left(\frac{200}{3}\right)}{2} = 1000 \\
15x + 500 = 1000 \\
15x = 500 \\
x = \frac{500}{15} = \frac{100}{3} = 33 \frac{1}{3}
\]

Therefore, the lengths of $AB$ and $DC$ are each $33\frac{1}{3}$ cm.
Solution 2

Let $P$, $Q$, $R$, and $S$ be the points of tangency on $AB$, $BC$, $CD$, and $DA$, respectively. Draw $OP$, $OQ$, $OR$, and $OS$. Then $OP = OQ = OR = OS = 15$ since they are each radii of the circle. Also, since a line drawn from the centre of the circle to a point of tangency is perpendicular to the tangent, $\angle OPA = \angle OQB = \angle ORD = \angle OSA = 90^\circ$.

Let $AB = x$, $AP = a$ and $DR = b$. Therefore, $DC = x$, $PB = x - a$ and $RC = x - b$. The following diagram shows all of the given and found information.

Join $A$ to $O$ forming two right triangles, $\triangle APO$ and $\triangle ASO$. Using the Pythagorean Theorem, $AP^2 = AO^2 - OP^2$ and $AS^2 = AO^2 - OS^2$. But $OP = OS$ since they are both radii. So the two expressions are equal and $AS = AP = a$ follows. Using exactly the same reasoning that was used to show $AP = AS = a$, we can show $DR = DS = b$, $BP = BQ = x - a$ and $CR = CQ = x - b$. This new information has been added to the diagram below.

We can now use the area of a trapezoid formula. As was shown in Solution 1, $SQ = SO + OQ$ is a height of the trapezoid. Therefore,

\[
\text{Area Trapezoid } ABCD = SQ \times (AD + BC) \div 2
\]

\[
1000 = (SO + OQ) \times ((AS + SD) + (BQ + QC)) \div 2
\]

\[
1000 = (15 + 15) \times ((a + b) + (x - a + x - b)) \div 2
\]

\[
1000 = (30) \times (2x) \div 2
\]

\[
1000 = 30x
\]

\[
\frac{100}{3} = x
\]

Therefore, the lengths of $AB$ and $DC$ are each $33\frac{1}{3}$ cm.

At the end of the statement of the problem, two facts were given. As an extension to this problem, the solver may wish to try to prove these two facts.