Problem of the Week
Problem C and Solution
Fill to 15

Problem
The game, “Fill to 15”, is a two-player game. The game board consists of 9 circles as shown in the diagram above. The players alternate turns placing discs numbered 1 to 9 in the circles on the board. Each number can only be used once in any game. The object of the game is to be the first player to place a disc so that the sum of the 3 numbers along a line through the centre circle is exactly 15. Alex and Blake play the game. Alex goes first. On her first move, Alex places a 6 in the centre circle. Then Blake places one of the eight remaining numbers in one of the empty circles on her first turn. Show that no matter which numbers Blake plays, Alex can win the game on either her second or third turn.

Solution
Since Alex played a 6 on her first turn, the other two discs in the line would need to add to 9 to make the total 15.

If, on her first turn, Blake plays one of the numbers 1, 2, 4, 5, 7 or 8, then there is an unused number that Alex can play on her second turn so that the sum of the line is 15. That is, if Blake plays a 1, then Alex will play an 8. (This is illustrated on the diagram to the right.) If Blake plays a 2, then Alex will play a 7. If Blake plays a 4, then Alex will play a 5. If Blake plays a 5, then Alex will play a 4. If Blake plays a 7, then Alex will play a 2. And if Blake plays an 8, then Alex will play a 1. In each of these 6 instances Alex can win on her second turn.

If, on her first turn, Blake places a 3 in any empty space, then the sum of the two discs in that line will be 9. Alex cannot win on her second turn since the only way to make the sum in that line 15 would be for her to play another 6. No number may be used more than once so this is not possible. However, if Alex completes the line by playing a 9 on her second turn, then the remaining discs will have numbers 1, 2, 4, 5, 7 and 8. Then, as we saw above, no matter what Blake plays on her second turn, there will be a number that Alex can place on that line so that the three numbers in the line add to 15. An example is illustrated in the diagram to the right.

Finally, if on her first turn, Blake places a 9 in any empty space, then the sum of the two discs on the line will already be 15 using just 2 discs. Alex cannot win on her second turn since playing any other disc in that line would make the sum greater than 15. However, if Alex completes the line by playing a 3 on her second turn, then the remaining discs will have numbers 1, 2, 4, 5, 7 and 8. Then, as we saw above, no matter what Blake plays next, there will be a number that Alex can place on that line so that the three numbers in the line add to 15. An example is illustrated in the diagram to the right.

Therefore, we have shown that no matter which numbers Blake plays, Alex can win the game on either her second or third turn.