26,13,40,...

Problem of the Week
Problem E and Solution
A Grand Sum

Problem
In a certain sequence, the first term is 26. If a term in this sequence is even, then the next term will be half the value of that term. If a term in this sequence is odd, then the next term in the sequence is one more than three times that term. By following these rules, the first three terms of this sequence are $26, 13$ and 40. If the sequence has n terms and the sum of these terms is a 4-digit number. How many different possible values of n are there?

Solution
If we continue the sequence using the given rules, we will get the following:

$$26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, \cdots \, .$$

After the first 8 terms, the sequence repeats the terms 4,2,1.

We now need to find the minimum and the maximum 4-digit sums that can be achieved by summing terms in the sequence.

The first possible 4-digit number is 1 000. Is it possible for the sum of the n terms to be exactly 1 000?

The sum of the first 8 terms is $26 + 13 + 40 + 20 + 10 + 5 + 16 + 8 = 138$. The sum of the repeating numbers is $4 + 2 + 1 = 7$. How many groups of 4,2,1 will be needed to sum to 1 000?

Let the number of groups be g. Then $7g + 138 = 1000$. Solving this gives $g = 123\frac{1}{7}$. Therefore, we cannot achieve the sum of 1 000 exactly.

However, there are 123 full repetition of 4,2,1 and the sum of these terms is $123(7) + 138 = 999$. This means the next number in the sequence will be 4 and the sum will be $999+4 = 1003$. This is the smallest 4-digit sum that a sequence could sum to. Therefore, the smallest sequence that has a 4-digit sum has $8 + 123(3) + 1 = 378$ terms.

Now we need to find the largest number of terms in a possible sequence. To do this we will solve $7g + 138 = 9999$ and get $g = 1408\frac{5}{7}$.

Now $1408(7) + 138 = 9994$. Again we have 1408 full repetitions, and the next number is 4 and the sum becomes $9994 + 4 = 9998$. The next number will be 2 and the sum becomes $9998 + 2 = 10000$ which has five digits. Therefore, the largest 4-digit sum is 9998 and the number of terms in this sequence is $8 + 1408(3) + 1 = 4233$.

Therefore the number of sequences with a 4-digit sum is $4233 - 378 + 1 = 3856$. Hence, there are 3856 different possible values for n.