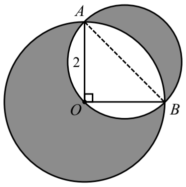


Problem of the Week

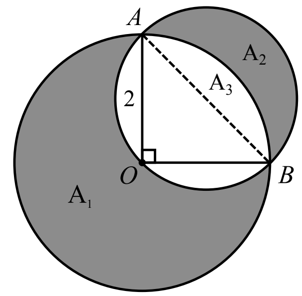
Problem E and Solution

Exclusive Areas

Problem

A and B lie on the circumference of the circle with centre O, radius 2, and $\angle AOB = 90^\circ$. Another circle, with diameter AB is drawn. O lies on the circumference of this second circle. The unshaded region is the area common to both circles. The shaded region is the area in one circle or the other circle but not in both. Determine the area of the shaded region in the diagram.

Solution



Label the region inside the larger circle but outside the smaller circle as A_1.
Label the region inside the smaller circle but outside the larger circle as A_2.
Label the region inside sector AOB but outside of $\triangle AOB$ as A_3.
We need to calculate $A_1 + A_2$.

First, we will calculate A_3.
Since $\angle AOB = 90^\circ$, the area of sector AOB is $\frac{90}{360} = \frac{1}{4}$ the area of the larger circle.
That is, the area of sector AOB is $\frac{1}{4} \times \pi(2)^2 = \pi$.
The area of $\triangle AOB$ is $\frac{1}{2}(OA)(OB) = \frac{1}{2}(2)(2) = 2$.
Therefore, $A_3 =$ area sector $AOB -$ area $\triangle AOB = (\pi - 2)$.

Next we will calculate A_2.
Since $\angle AOB = 90^\circ$, the Pythagorean theorem tells us $AB^2 = OA^2 + OB^2 = 2^2 + 2^2 = 8$.
Therefore, $AB = \sqrt{8} = 2\sqrt{2}$, since $AB > 0$.
Since AB is a diameter of the smaller circle, the radius is $\frac{1}{2}AB = \frac{1}{2}\left(2\sqrt{2}\right) = \sqrt{2}$.
Therefore, $A_2 + A_3 = \frac{1}{2}$(the area of the circle with radius $\sqrt{2}$) $= \frac{1}{2}\pi(\sqrt{2})^2 = \frac{1}{2}\pi(2) = \pi$.
Therefore, $A_2 = \pi - A_3 = \pi - (\pi - 2) = 2$.

Finally, we will calculate A_1.
A_1 represents the area inside the larger circle which is not in the smaller circle.
The larger circle has radius 2 and area $\pi(2)^2 = 4\pi$.
The smaller circle has radius $\sqrt{2}$ and area $\pi(\sqrt{2})^2 = 2\pi$.

$$\begin{aligned} A_1 &= (\text{area larger circle}) - \frac{1}{2}(\text{area smaller circle}) - A_3 \\ &= 4\pi - \frac{1}{2}(2\pi) - (\pi - 2) \\ &= 4\pi - \pi - \pi + 2 \\ &= 2\pi + 2 \end{aligned}$$

Therefore, the area of the shaded region $A_1 + A_2 = (2\pi + 2) + 2 = (2\pi + 4)$ units^2.