$$14\overline{)20000}\quad 1428\ \text{R}8$$

Problem of the Week
Problem E and Solution
Remainders

Problem

One day Gil and Aditi were doing some dividing. They noticed that when 20 000 is divided by 14 the remainder is 8. Gil further noted that when 20 000 is divided by 21, the remainder is also 8. Aditi noted that when 20 000 is divided by 34, the remainder is once again 8. How many positive five-digit integers have the same remainder (not necessarily 8) when divided by 14, 21, and 34?

Solution

Since $14 = 2 \times 7, 21 = 3 \times 7$ and $34 = 2 \times 17$, the least common multiple (LCM) of 14, 21, and 34 is LCM(14,21,34) $= 2 \times 3 \times 7 \times 17 = 714$.

Suppose n is a positive integer. The following statements are true:
Every integer of the form $714n$ will have a remainder of 0 when divided by 14, 21, and 34.
Every integer of the form $714n + 1$ will have a remainder of 1 when divided by 14, 21, and 34.
Every integer of the form $714n + 2$ will have a remainder of 2 when divided by 14, 21, and 34.
Every integer of the form $714n + 3$ will have a remainder of 3 when divided by 14, 21, and 34.
$\vdots$
Every integer of the form $714n + 12$ will have a remainder of 12 when divided by 14, 21, and 34.
Every integer of the form $714n + 13$ will have a remainder of 13 when divided by 14, 21, and 34.

However, every integer of the form $714n + 14$ will not have the same remainder when divided by 14, 21, and 34. The remainder will be 0, 14, and 14 respectively. Therefore, we need to find the number of five-digit integers that have the form $714n + r$ where $0 \leq r \leq 13$.

The smallest five-digit integer that is a multiple of 714 can be found by dividing 10 000 by 714. Since $\frac{10\,000}{714} \approx 14.0056$, the first five-digit multiple is $714 \times 15 = 10\,710$. This means the integers from 10 710 to 10 723 (10 710 + 13), have the same remainder when divided by 14, 21, and 34.

To find the largest five-digit integer that is a multiple of 714 we divide 100 000 by 714. Since $\frac{100\,000}{714} \approx 140.056$, the largest five-digit multiple is $714 \times 140 = 99\,960$. This means the integers from 99 960 to 99 973 (99 960 + 13), have the same remainder when divided by 14, 21, and 34.

Thus, $714n$ is a positive five-digit integer for $15 \leq n \leq 140$. The number of positive five-digit integers that are divisible by 714 is $140 - 15 + 1 = 126$. For each of multiples of 714, there are 14 integers that have the same remainder when we divide by 14, 21, and 34. This gives a total of $126 \times 14 = 1764$ integers that have the same remainder when divided by 14, 21, and 34.

However, we need to check near 10 000. The largest multiple of 714 that is less than 10 000 is $714 \times 14 = 9996$. This means the integers between 9996 and 10 009 (9996 + 13) have the same remainder when they are divided by 14, 21, and 34. Of these, 10 are five-digit integers.

We should also check integers near 100 000. We found out the largest multiple of 714 less that 100 000 is 99 960. If we increase this by 13, we get 99 973 which is also less than 100 000. All of these multiples have already been counted.

Therefore, the number of positive five-digit integers that leave the same remainder when divided 14, 21, and 34 is $1764 + 10 = 1774$.