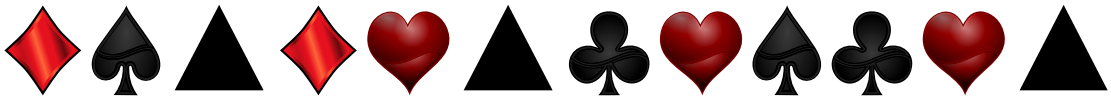
Problem of the Week
Problem E and Solution
Symbolic Sequence

Problem

A sequence is formed using the following symbols: a club ♣, a diamond ♦, a heart ♥, a spade ♠, and a triangle ▲. Symbols can be repeated and are placed in the sequence according to the following rules: the sequence contains a total of 12 symbols; the sequence contains exactly 3 triangles located in positions 3, 6 and 12; a club must be followed by a diamond or a heart; a diamond must be followed by a heart or a spade; a heart must be followed by a spade or a triangle; a spade must be followed by a triangle or a club; and a triangle must be followed by a club or a diamond. The following is an example of a valid sequence.



How many different valid sequences are there?

Solution

In the solution we will use a C to represent a club ♣, a D to represent a diamond ♦, an H to represent a heart ♥, an S to represent a spade ♠, and a T to represent a triangle ▲.

We need to determine how many possible ways to fill the first two positions. We will multiply this by the number of ways to fill positions 4 and 5. We will then multiply this by the number of ways to fill positions 7 to 11.

- Filling Positions 1 and 2

  Since there is a T in position 3, we know that position 2 must contain an H or an S. If position 2 is an H, then position 1 can be a C or a D. If position 2 is an S, then position 1 can be a D or an H. There are 4 possible ways to start the sequence, namely CH, DH, DS and HS.

- Filling Positions 5 and 6

  Since there is a T in position 3, we know that position 4 must contain a C or a D. Since there is a T in position 6, we know position 5 must be an H or an S. This gives 4 possibilities for positions 4 and 5, namely CH, CS, DH, DS. Of these, CS is invalid since C must be followed by a D or H. Therefore, there are 3 combinations that work, namely CH, DH and DS.

Therefore, to fill positions 1,2,4 and 5, there are $4 \times 3 = 12$ valid possibilities.

When filling positions 7 to 11 it is important to note that no T's are permitted in any of these positions since the sequence only has 3 T's and we know where they are. Also, position 7 must be a C or a D, and position 11 must be an H or an S. We will proceed with two cases.

- Filling Positions 7 to 11 with Position 7 containing a C

  If position 7 is a C, then we can have CD or CH.

  From here we can have CDH, CDS or CHS (CHT is invalid).

  From here we can have CDHS, CDSC, CHSC (CDHT, CDST and CHST are invalid).

  From here we can have CDSCH and CHSCH (CDHST, CDHSC, CDSCD and CHSCD are not valid, since there must be an H or an S in position 11).

  Therefore, there are 2 ways to fill positions 7 to 11 with position 7 containing a C, namely CDSCH and CHSCH.

- Filling Positions 7 to 11 with Position 7 containing a D

  If position 7 is a D, then we can have DH or DS.

  From here we can have DHS or DSC (DHT and DST are invalid).

  From here we can have DHSC, DSCD and DSCH (DHST is invalid).

  From here we can have DHSCH, DSCDH, DSCDS or DSCHS (DHSCD and DSCHT are invalid, since there must be an H or an S in position 11).

  There are 4 ways to fill positions 7 to 11 with position 7 containing a D, namely DHSCH, DSCDH, DSCDS and DSCHS.

In total there are $2 + 4 = 6$ ways to fill positions 7 to 11.

For each of the 6 ways to fill positions 7 to 11, there are 12 ways to fill positions 1,2,4 and 5. Therefore, there are a total of $6 \times 12 = 72$ valid sequences.