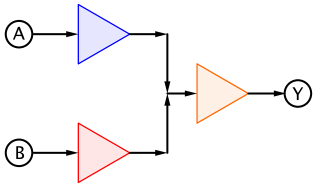

Problem of the Week

Problem E and Solution

How Small Can It Get?

Problem

A computer program takes in two inputs, InputⒶ and InputⒷ, and produces one Final OutputⓎ as follows. InputⒶ is doubled, then the result is squared. This result is then reduced by 4 times InputⒶ. The result is OutputⒶ. InputⒷ is squared, then the result is increased by 6 times InputⒷ. The result is OutputⒷ.

The program produces a Final OutputⓎ = OutputⒶ + OutputⒷ. Determine the minimum final output and the two input values which produce this minimum.

Solution

In order to minimize the final output, we need to minimize both OutputⒶ and OutputⒷ.

First, let's minimize OutputⒶ. Let x be InputⒶ.

The computer program doubles x to get $2x$. It squares this result to get $(2x)^2 = 4x^2$. It then reduces this number by 4 times InputⒶ to get OutputⒶ $= 4x^2 - 4x$.
So we need to minimize $4x^2 - 4x$. This is a quadratic and so represents a parabola. Since the coefficient of x^2 is positive, it is concave up and so its minimum value occurs at its vertex. We can find the vertex by completing the square.

$$4x^2 - 4x = 4(x^2 - x) = 4\left(x^2 - x + \frac{1}{4} - \frac{1}{4}\right) = 4\left(x^2 - x + \frac{1}{4}\right) - 1 = 4\left(x - \frac{1}{2}\right)^2 - 1$$

The vertex is at $\left(\frac{1}{2}, -1\right)$, and so the minimum value of $4x^2 - 4x$ is -1 and occurs when $x = \frac{1}{2}$.
Therefore, the minimum value of OutputⒶ is -1 and occurs when InputⒶ is $\frac{1}{2}$.

Now let's minimize OutputⒷ. Let y be InputⒷ.

The program squares y to get y^2. It then increases the result by 6 times InputⒷ to get $y^2 + 6y$.
So we need to minimize $y^2 + 6y$. This is a quadratic and so represents a parabola. Since the coefficient of y^2 is positive, it is concave up and so its minimum value occurs at its vertex. We can find the vertex by completing the square.

$$y^2 + 6y = (y^2 + 6y + 9) - 9 = (y + 3)^2 - 9$$

The vertex is at $(-3, -9)$, and so the minimum of $y^2 + 6y$ is -9 and occurs when $y = -3$.
Therefore, the minimum value of OutputⒷ is -9 and occurs when InputⒷ is -3.

Therefore, the minimum value of the Final OutputⓎ is $-1 + (-9) = -10$ and occurs when InputⒶ is $\frac{1}{2}$ and InputⒷ is -3.

Aside: This problem essentially asked us to minimize the multivariable function
$f(x, y) = 4x^2 + y^2 - 4x + 6y$.