Problem of the Week

Problem E and Solution

Seeing Perfectly

Problem

How many divisors of 2020^{2020} are perfect squares?

Solution

First, let's look at the prime factorization of four perfect squares:

$$9 = 3^2,\ 16 = 2^4,\ 36 = 2^2 \times 3^2,\ \text{and } 129\,600 = 2^6 \times 3^4 \times 5^2.$$

Note that, in each case, the exponent on each of the prime factors is even. For some integer a, if m is an even integer greater than or equal to zero then a^m is a perfect square.

$$\begin{aligned} \text{Now } 2020^{2020} &= (2^2 \times 5 \times 101)^{2020} \\ &= (2^2)^{2020} \times 5^{2020} \times 101^{2020} \\ &= 2^{4040} \times 5^{2020} \times 101^{2020} \end{aligned}$$

All positive divisors of 2020^{2020} will be of the form $2^k \times 5^n \times 101^p$, $0 \leq k \leq 4040$, $0 \leq n \leq 2020$, and $0 \leq p \leq 2020$, where k, n and p are each integers.

For 2^k to be a perfect square, k must be an even integer such that $0 \leq k \leq 4040$. There are $4040 \div 2 = 2020$ even numbers from 1 to 4040. The number 0 is also even so there are 2021 values of k such that 2^k is a perfect square.

For 5^n to be a perfect square, n must be an even integer such that $0 \leq n \leq 2020$. There are 1010 even numbers from 1 to 2020. Zero is also even so there are 1011 values of n such that 5^n is a perfect square.

Similarly, for 101^p to be a perfect square, p must be an even integer such that $0 \leq p \leq 2020$. There are 1010 even numbers from 1 to 2020. Zero is also even so there are 1011 values of p such that 101^p is a perfect square.

For each of the 2021 values of k, there are 1011 values of n and 1011 values of p so there are $2021 \times 1011 \times 1011 = 2\ 065\ 706\ 541$ perfect square divisors of 2020^{2020}.

$\therefore 2020^{2020}$ has 2 065 706 541 perfect square divisors. This is over 2 billion perfect square divisors!

For a smaller number like 12^6 we could also determine the number of perfect square divisors using the above approach. We could verify that the approach is valid by physically listing all of the divisors and then counting the number that are perfect squares. The number 12^6 has "only" 91 positive divisors, 28 of which are perfect squares. It is not practical to list all the divisors of 2020^{2020} and then count the perfect squares. But our approach allows us to still count the number of perfect square divisors.