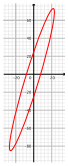

Problem of the Week
Problem E and Solution
Elliptical Workout

Problem

When we graph the equation $(x+1)^2+(y-2)^2=100$, we get a circle centred at the point $(-1,2)$ with a radius of 10. When we graph $10x^2-6xy+4x+y^2=621$, we get a shape known as an ellipse. This is the graph shown above.

List all the order pairs (x,y) of non-negative integers x and y that satisfy the equation $10x^2-6xy+4x+y^2=621$.

Solution

Starting with the given equation, we obtain equivalent equations:

$$\begin{aligned} 10x^2-6xy+4x+y^2 &= 621 \\ 9x^2-6xy+y^2+x^2+4x &= 621 \\ 9x^2-6xy+y^2+x^2+4x+4 &= 621+4 \\ (3x-y)^2+(x+2)^2 &= 625 \end{aligned}$$

Notice that $625=25^2$.

Since x and y are both integers, then the left side of the given equation is the sum of two perfect squares. Since any perfect square is non-negative, then each of these perfect squares is at most $625=25^2$.

The perfect squares from 0^2 and 25^2 are:

0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625

The pairs of perfect squares that sum to 625 are $625+0$, $576+49$, and $400+225$.

Therefore, $(3x-y)^2$ and $(x+2)^2$ equal 25^2 and 0^2 in some order, or 24^2 and 7^2 in some order, or 20^2 and 15^2 in some order.

Furthermore, $3x-y$ and $x+2$ equal ±25 and ±0 in some order, or ±24 and ±7 in some order, or ±20 and ±15 in some order.

Since $x\geq0$ then $(x+2)\geq2$, so we need to consider the possibilities where $(x+2)=25,24,7,20,15$.

We will use a table to find the values for x and y.

$x+2$	x	$3x-y$	y	$x \geq 0$ and $y \geq 0$?
25	23	0	69	yes
24	22	7	59	yes
24	22	-7	73	yes
7	5	24	-9	no
7	5	-24	39	yes
20	18	15	39	yes
20	18	-15	69	yes
15	13	20	19	yes
15	13	-20	59	yes

Therefore the ordered pairs are
$(23, 69), (22, 59), (22, 73), (5, 39), (18, 39), (18, 69), (13, 19), (13, 59)$.