Problem of the Week
Problem E and Solution
One Thousand Zeros

Problem

The product of the integers from 1 to n can be written in abbreviated form as $n!$ and we say "n *factorial*". So $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$.
For example, $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$, and

$$11! = 11 \times 10 \times 9 \times \ldots \times 3 \times 2 \times 1 = 39\,916\,800.$$

Note that $6!$ ends in 1 zero and $11!$ ends in 2 zeros.
Determine the smallest possible value of n such that $n!$ ends in exactly 1000 zeros.

Solution

When finding a solution to this problem, it may be helpful to work with possible values for n to determine the number of zeros that $n!$ ends in. One could use a calculator as part of this but many calculators switch to scientific notation around $14!$. A trial and error approach could work but it may be very time consuming. Our approach will be very systematic.

A zero is added to the end of a number when we multiply by 10. Multiplying a number by 10 is the same as multiplying a number by 2 and then by 5, or by 5 and then by 2, since $2 \times 5 = 10$ and $5 \times 2 = 10$.

So we want n to be the smallest integer such that the factorization of $n!$ contains 1000 5s and 1000 2s. Every even number has a 2 in its factorization and every number that is a multiple of 5 has a 5 in its factorization. There are more numbers less than or equal to n that are multiples of 2 than multiples of 5. So once we find a number n such that $n!$ has 1000 5s in its factorization, we can stop, we know that there will be a sufficient number of 2s in its factorization.

There are $\left\lfloor \frac{n}{5} \right\rfloor$ numbers less than or equal to n that are divisible by 5. Note, the notation $\lfloor x \rfloor$ means *the floor of* x and is the largest integer less than or equal to x. So $\lfloor 4.2 \rfloor = 4$, $\lfloor 4.9 \rfloor = 4$ and $\lfloor 4 \rfloor = 4$. Also, since $5 \times 1000 = 5000$, we know that $n \leq 5000$.

Numbers that are a multiple of 25 will add an additional factor of 5, since $25 = 5 \times 5$.
There are $\left\lfloor \frac{n}{25} \right\rfloor$ numbers less than or equal to n that are divisible by 25.

Numbers that are a multiple of 125 will add an additional factor of 5, since $125 = 5 \times 5 \times 5$ and two of the factors have already been counted when we looked at 5 and 25.
There are $\left\lfloor \frac{n}{125} \right\rfloor$ numbers less than or equal to n that are divisible by 125.

Numbers that are a multiple of 625 will add an additional factor of 5, since $625 = 5 \times 5 \times 5 \times 5$ and three of the factors have already been counted when we looked at 5, 25 and 125.
There are $\left\lfloor \frac{n}{625} \right\rfloor$ numbers less than or equal to n that are divisible by 625.

Numbers that are a multiple of 3125 will add an additional factor of 5, since $3125 = 5^5$ and four of the factors have already been counted when we looked at 5, 25, 125 and 625.
There are $\left\lfloor \frac{n}{3125} \right\rfloor$ numbers less than or equal to n that are divisible by 3125.

The next power of 5 to consider is $5^6 = 15\,625$. But since $n \leq 5000$, we can stop.

So we know that n must satisfy

$$\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor + \left\lfloor \frac{n}{125} \right\rfloor + \left\lfloor \frac{n}{625} \right\rfloor + \left\lfloor \frac{n}{3125} \right\rfloor = 1000$$

Let's ignore the floor function. We know that n is going to be *close* to satisfying

$$\begin{aligned}
\frac{n}{5} + \frac{n}{25} + \frac{n}{125} + \frac{n}{625} + \frac{n}{3125} &= 1000 \\
\frac{625n}{3125} + \frac{125n}{3125} + \frac{25n}{3125} + \frac{5n}{3125} + \frac{n}{3125} &= 1000 \\
\frac{781}{3125}n &= 1000 \\
n &= \frac{1000 \times 3125}{781} \\
n &= 4001.2
\end{aligned}$$

How many zeros are at the end of 4001! ?

$$\begin{aligned}
& \left\lfloor \frac{4001}{5} \right\rfloor + \left\lfloor \frac{4001}{25} \right\rfloor + \left\lfloor \frac{4001}{125} \right\rfloor + \left\lfloor \frac{4001}{625} \right\rfloor + \left\lfloor \frac{4001}{3125} \right\rfloor \\
= \; & \lfloor 800.2 \rfloor + \lfloor 160.04 \rfloor + \lfloor 32.008 \rfloor + \lfloor 6.401\,6 \rfloor + \lfloor 1.280\,32 \rfloor \\
= \; & 800 + 160 + 32 + 6 + 1 \\
= \; & 999
\end{aligned}$$

So the factorization of 4001! has 999 zeros at the end. We need 1 more factor of 5 in order to have 1000 zeros at the end. The first integer after 4001 to contain a factor of 5 is 4005.

Therefore, 4005 is the smallest number such that 4005! ends in 1000 zeros.

Indeed, we can check. The number of zeros at the end of 4004! is equal to the number of 5's in its factorization, which is equal to

$$\begin{aligned}
& \left\lfloor \frac{4004}{5} \right\rfloor + \left\lfloor \frac{4004}{25} \right\rfloor + \left\lfloor \frac{4004}{125} \right\rfloor + \left\lfloor \frac{4004}{625} \right\rfloor + \left\lfloor \frac{4004}{3125} \right\rfloor \\
= \; & \lfloor 800.8 \rfloor + \lfloor 160.16 \rfloor + \lfloor 32.032 \rfloor + \lfloor 6.406\,4 \rfloor + \lfloor 1.281\,28 \rfloor \\
= \; & 800 + 160 + 32 + 6 + 1 \\
= \; & 999
\end{aligned}$$

The number of zeros at the end of 4005! is equal to the number of 5's in its factorization, which is equal to

$$\begin{aligned}
& \left\lfloor \frac{4005}{5} \right\rfloor + \left\lfloor \frac{4005}{25} \right\rfloor + \left\lfloor \frac{4005}{125} \right\rfloor + \left\lfloor \frac{4005}{625} \right\rfloor + \left\lfloor \frac{4005}{3125} \right\rfloor \\
= \; & \lfloor 801 \rfloor + \lfloor 160.2 \rfloor + \lfloor 32.04 \rfloor + \lfloor 6.408 \rfloor + \lfloor 1.281\,6 \rfloor \\
= \; & 801 + 160 + 32 + 6 + 1 \\
= \; & 1000
\end{aligned}$$