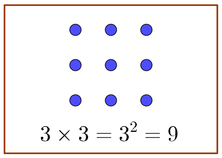
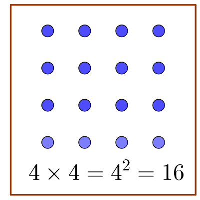


Problem of the Week

Problem D and Solution

This Number Makes It Perfect



Problem

The positive even integers 2 to 1600, inclusive, are each multiplied by the same positive integer, n. All of the products are then added together and the resulting sum is a perfect square.

Determine the value of the smallest positive integer n that makes this true.

Solution

What does the prime factorization of a perfect square look like? Let's look at a few examples: $9 = 3^2$, $36 = 6^2 = 2^2 3^2$, and $129\,600 = 360^2 = 2^6 5^2 3^4$. Notice that the exponent on each of the prime factors in the prime factorization in each of the three examples is an even number.

The positive integer n is the smallest positive integer such that

$$2n + 4n + 6n + \cdots + 1596n + 1598n + 1600n \qquad (1)$$

is a perfect square.

Factoring (1), we obtain

$$\begin{aligned}
& 2n(1 + 2 + 3 + \cdots + 798 + 799 + 800) \\
= \;& 2n\left(\frac{800 \times 801}{2}\right) \\
= \;& n(800)(801) && (2) \\
= \;& n[(2)(2)(2)(2)(2)(5)(5)][(3)(3)(89)] \\
= \;& n(2^5)(5^2)(3^2)(89) && (3)
\end{aligned}$$

In going from (2) to (3), we have expressed the 800×801 as the product of prime factors.

We need to determine what additional factors are required to make the quantity in (3) a perfect square such that n is as small as possible. In order for the exponent on each prime in the prime factorization to be even, we need n to be $89 \times 2 = 178$. Then the quantity in (3) becomes

$$n(2^5)(5^2)(3^2)(89) = (2)(89)(2^5)(5^2)(3^2)(89) = (2^6)(5^2)(3^2)(89^2) = \left[(2^3)(5)(3)(89)\right]^2,$$

a perfect square.

Therefore, the smallest positive integer is 178 and the perfect square is

$$178 \times 800 \times 801 = 114\,062\,400 = (10\,680)^2.$$