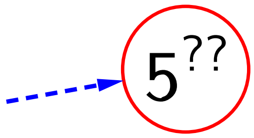

Problem of the Week
Problem D and Solution
The Power of Five

Problem

The number N is the product of the first 1000 positive integers and can be written as 1000!. We say, "1000 *factorial*." That is, $N = 1000! = 1000 \times 999 \times 998 \times 997 \times \cdots \times 3 \times 2 \times 1$. N is divisible by $5, 25, 125, 625, \cdots$. Each of these factors is a power of 5. That is, $5 = 5^1,\ 25 = 5^2,\ 125 = 5^3,\ 625 = 5^4,$ and so on. Determine the largest power of 5 that divides N.

Solution

Solution 1

In order to determine the largest power of 5 that divides N, we need to count the number of times the factor 5 appears in the factorization of N.

First, let's look at the numbers that are divisible by 5 in N!. Each of the numbers $\{5, 10, 15, 20, \ldots, 990, 995, 1000\}$ contains a factor of 5. That is a total of $\frac{1000}{5} = 200$ factors of 5.

Those numbers that are multiples of 25 will add an additional factor of 5, since $25 = 5 \times 5$. There are $\frac{1000}{25} = 40$ numbers less than or equal to 1000 which are multiples of 25. So we gain another 40 factors of 5 bringing the total to $200 + 40 = 240$.

Those numbers that are multiples of 125 will add an additional factor of 5. This is because $125 = 5 \times 5 \times 5$ and two of the factors have already been counted when we looked at 5 and 25. There are $\frac{1000}{125} = 8$ numbers less than or equal to 1000 which are multiples of 125. So we gain another 8 factors of 5 bringing the total to $240 + 8 = 248$.

Those numbers that are multiples of 625 will add an additional factor of 5. This is because $625 = 5 \times 5 \times 5 \times 5$ and three of the factors have already been counted when we looked at 5, 25 and 125. There is 1 number less than 1000 which is a multiple of 625 (namely, 625). So we gain another factor of 5 bringing the total to $248 + 1 = 249$. So, when N is factored there are 249 factors of 5. Therefore, the largest power of 5 that divides N is 5^{249}.

Solution 2

There are many similarities between solution 1 and the following solution. In this solution we will divide out multiples of 5 until there are none left.

1. In 1000!, there are $\frac{1000}{5} = 200$ multiples of 5, namely, $\{5, 10, 15, \cdots, 990, 995, 1000\}$. No other numbers from 1 to 1000 are divisible by 5. If we divide each number in this first list by 5, we obtain the second list $\{1, 2, 3, \cdots, 198, 199, 200\}$.
2. This second list contains $\frac{200}{5} = 40$ more multiples of 5, namely, $\{5, 10, 15, \cdots, 190, 195, 200\}$. No other numbers from 1 to 200 are divisible by 5. If we divide each number in this second list by 5, we obtain the third list $\{1, 2, 3, \cdots, 38, 39, 40\}$.
3. This third list contains $\frac{40}{5} = 8$ more multiples of 5, namely, $\{5, 10, 15, 20, 25, 30, 35, 40\}$. No other numbers from 1 to 40 are divisible by 5. If we divide each number in the third list by 5, we obtain the fourth list $\{1, 2, 3, 4, 5, 6, 7, 8\}$.
4. This fourth list contains 1 more multiples of 5, namely the number 5. No other numbers from 1 to 8 are divisible by 5.

In total, there are $200 + 40 + 8 + 1 = 249$ factors of 5 in 1000!. Therefore, the largest power of 5 that divides N is 5^{249}.

An interpretation of what has happened is in order. When we created the first list of multiples of 5, we discovered that there were 200 numbers from 1 to 1000 that were divisible by 5. When we created the second list of multiples of 5, we were actually counting the 40 numbers from 1 to 1000 that were divisible by 25. When we created the third list of multiples of 5, we were actually counting the 8 numbers from 1 to 1000 that were divisible by 125. And finally, when we created the fourth list of multiples of 5, we were actually counting the 1 number from 1 to 1000 that was divisible by 625.

NOTE: We define $\text{int}(n)$ as the largest integer less than or equal to n. For example, int(42)=42 and int(37.6)=37. We can count the number of numbers divisible by various powers of 5 as follows:

$$\begin{aligned}
\#\text{ of Factors of 5 in 1000!} &= \text{int}\left(\frac{1000}{5}\right) + \text{int}\left(\frac{1000}{25}\right) + \text{int}\left(\frac{1000}{125}\right) + \text{int}\left(\frac{1000}{625}\right) \\
&= \text{int}(200) + \text{int}(40) + \text{int}(8) + \text{int}(1.6) \\
&= 200 + 40 + 8 + 1 \\
&= 249
\end{aligned}$$