Problem of the Week
Problem D and Solution
A Different Difference

Problem

The *non-negative difference* between two numbers a and b is $b-a$ or $a-b$, whichever is greater than or equal to zero. For example, the non-negative difference between 24 and 64 is 40. In a sequence that begins $74, 60, 14, 46, 32, \cdots$, each number after the second number is obtained by finding the non-negative difference between the previous 2 numbers. Determine the sum of the first 1300 numbers in the sequence.

Solution

We will start by generating more terms of the sequence in an attempt to find a pattern.

Using the rule for creating the sequence, we obtain

$$74, 60, 14, 46, 32, 14, 18, 4, 14, 10, 4, 6, 2, 4, 2, 2, 0, 2, 2, 0, 2, 2, 0, \cdots .$$

The first 14 terms of the sequence have no apparent pattern. The values of the 15^{th}, 16^{th} and 17^{th} terms repeat as the sequence is extended. So the 15^{th} term, 18^{th} term, 21^{st} term, and so on, all equal the first 2 in the string 2,2,0.

The first 14 terms of the sequence are followed by n groups of 2,2,0. What is the value of n? We want a total of 1300 terms. If we remove the first 14 terms, we require $1300 - 14 = 1286$ more terms. If we divide 1286 by 3 we are able to determine how many complete strings of 2,2,0 are needed. The result is $1286 \div 3 = 428\frac{2}{3}$. This tells us that we need 428 complete copies of 2,2,0 and $\frac{2}{3}$ of a copy of 2,2,0, namely 2,2.

The required sum is the sum of the first 14 terms plus $428 \times (2+2+0) + 2 + 2$. The sum of the first 14 terms is 302. The sum of the first 1300 terms of the sequence is $302 + 428 \times 4 + 4 = 2018$.

The sum of the first 1300 terms of the sequence is 2018.