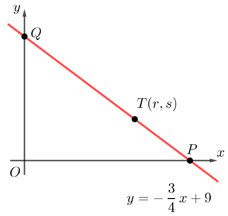



# Problem of the Week

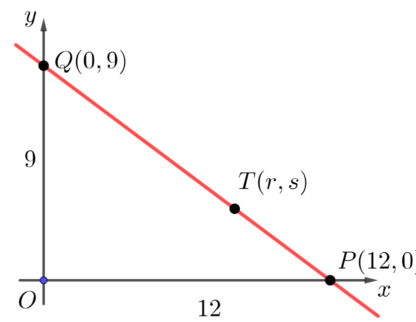
## Problem D and Solution

## A Point of Division

### Problem

The line $y = -\frac{3}{4}x + 9$ crosses the $x$-axis at $P$ and the $y$-axis at $Q$. Point $T(r, s)$ lies on the line segment $PQ$ such that the area of $\triangle POQ$ is three times the area of $\triangle TOP$. Determine the values of $r$ and $s$, the coordinates of $T$.

### Solution

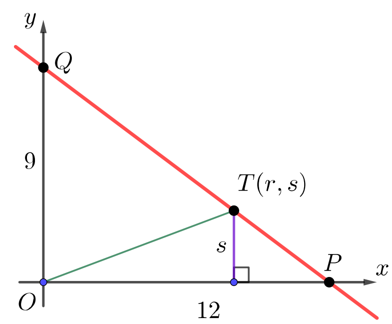



We begin by calculating the coordinates of $P$ and $Q$, the $x$- and $y$-intercepts of the line $y = -\frac{3}{4}x + 9$.

Since the equation of the line is written in the form $y = mx + b$ where $b$ is the $y$-intercept of the line, the $y$-intercept is 9 and so the coordinates of $Q$ are $(0, 9)$. To determine the $x$-intercept, set $y = 0$ to obtain $0 = -\frac{3}{4}x + 9 \implies \frac{3}{4}x = 9 \implies x = 12$. Thus, $P$ has coordinates $(12, 0)$.

We now present two different solutions to the problem.

### Solution 1




Since $\triangle POQ$ is a right triangle with base $PO = 12$ and height $OQ = 9$, using the formula area $= \frac{\text{base}\times\text{height}}{2}$, we have area$(\triangle POQ) = \frac{12\times 9}{2} = 54$.

Since the area of $\triangle POQ$ is three times the area of $\triangle TOP$, area$(\triangle TOP) = \frac{1}{3}(\text{area}(\triangle POQ)) = \frac{1}{3}(54) = 18$.

$\triangle TOP$ has area 18, base $PO = 12$ and height $s$. Using the formula area $= \frac{\text{base}\times\text{height}}{2}$, we have

$$\begin{aligned} \text{area}(\triangle TOP) &= \frac{PO \times s}{2} \\ 18 &= \frac{12 \times s}{2} \\ 18 &= 6s \\ \therefore\ s &= 3 \end{aligned}$$

$T(r, s)$ lies on the line $y = -\frac{3}{4}x + 9$ and $s = 3$ so we can substitue $x = r$ and $y = 3$

$$\begin{aligned} 3 &= -\frac{3}{4}r + 9 \\ \frac{3}{4}r &= 6 \\ \therefore\ r &= 8 \end{aligned}$$

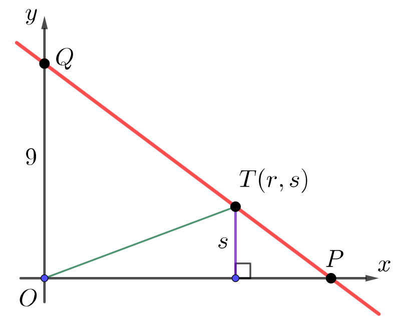
Therefore, $T$ is the point $(r, s) = (8, 3)$.

## Solution 2

If two triangles have equal bases, the areas of the triangles are proportional to the heights of the triangles.

$\triangle POQ$ and $\triangle TOP$ have the same base, $OP$.

Since the area of $\triangle POQ$ is 3 times the area of $\triangle TOP$, then the height of $\triangle POQ$ is 3 times the height of $\triangle TOP$. In other words, the height of $\triangle TOP$ is $\frac{1}{3}$ the height of $\triangle POQ$. $\triangle POQ$ has height $OQ = 9$ and $\triangle TOP$ has height $s$. Therefore, $s = \frac{1}{3}(OQ) = \frac{1}{3}(9) = 3$.




Since $T(r, s)$ lies on the line $y = -\frac{3}{4}x + 9$, we have

$$\begin{aligned} s &= -\frac{3}{4}r + 9 \\ 3 &= -\frac{3}{4}r + 9 \\ \frac{3}{4}r &= 6 \\ \therefore\ r &= 8 \end{aligned}$$

Therefore, $T$ is the point $(r, s) = (8, 3)$.

Note that it was actually unnecessary to find the $x$-intercept for the second solution as it was never used in the second solution.

**For Further Thought:**

Find the coordinates of $S$, another point on line segment $QP$, so that

$$\text{the area of } \triangle SOQ = \text{ the area of } \triangle TOP,$$

thus creating three triangles of equal area. How are the points $Q$, $S$, $T$, and $P$ related?