Problem of the Week
Problem C and Solution
000000 Means the End

Problem

The product of the positive integers 1 to 6 is $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.
and can be written in an abbreviated form as 6!. We say, "6 *factorial*". So $6! = 720$.

The product of the positive integers from 1 to 12 is

$$12 \times 11 \times 10 \times \ldots \times 3 \times 2 \times 1 = 479\,001\,600$$

and can be written in an abbreviated form as 12!. We say, "12 *factorial*". The $\cdots$ represents the product of all of the missing integers between 10 and 3.

For a positive integer n, the product of the positive integers from 1 to n is $n!$.

Find the smallest possible value of n such that $n!$ ends in exactly six zeroes.

Solution

We start by examining the first few factorials:

$$\begin{aligned}
1! &= 1\\
2! &= 2 \times 1 = 2\\
3! &= 3 \times 2 \times 1 = 6\\
4! &= 4 \times 3 \times 2 \times 1 = 24\\
5! &= 5 \times 4 \times 3 \times 2 \times 1 = \mathbf{120}\\
6! &= 6 \times (5 \times 4 \times 3 \times 2 \times 1) = 6 \times 5! = 6(120) = 720\\
7! &= 7 \times (6 \times 5 \times 4 \times 3 \times 2 \times 1) = 7 \times 6! = 7(720) = 5040\\
8! &= 8 \times (7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 8 \times 7! = 8(5040) = 40\,320\\
9! &= 9 \times (8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 9 \times 8! = 9(40\,320) = 362\,880\\
10! &= 10 \times (9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 10 \times 9! = 10(362\,880) = \mathbf{3\,628\,800}
\end{aligned}$$

These numbers are getting very large and soon will not fit on the display of a standard calculator. So, let's look at what is going on.

We observe that 5! ends in 0 and 10! ends in 00. Notice that the number of zeros at the end of the number increased by one at each of 5! and at 10!. Why is this?

A zero is added to the end of a number when we multiply by 10. Multiplying a number by 10 is the same as multiplying a number by 2 and then by 5, or by 5 and then by 2, since $2 \times 5 = 10$ and $5 \times 2 = 10$. We must determine the next time we multiply by 2 and 5 (in some order), to know the next time the number of zeros at the end of the number increases again. Every time we multiply by an even number we are multiplying by at least one more 2. There are less multiples of 5. Each multiple of 5 will affect the number of zeros at the end of the product.

Multiplying by 11, 12, 13, and 14 increases the number of 2s we multiply by but not the number of 5s. So the number of zeros at the end of the product does not change. The next time we multiply by a 5 is when we multiply by 15 since $15 = 5 \times 3$. So 15! will end in exactly three zeros, 000.

Multiplying by 16, 17, 18, and 19 increases the number of 2's we multiply by but not the number of 5s. So the number of zeros at the end of the product does not change. The next time we multiply by a 5 is when we multiply by 20 since $20 = 4 \times 5$. So 20! will end in exactly four zeros, 0000.

Multiplying by 21, 22, 23, and 24 increases the number of 2s we multiply by but not the number of 5s. The next time we multiply by a 5 is when we multiply by 25. In fact, multiplying by 25 is the same as multiplying by 5 twice since $25 = 5 \times 5$. So when we multiply by 25, we will increase the number of zeros on the end of the product by two. So 25! will end in exactly six zeros, 000 000.

Then, for $n!$ to end in six zeros, the smallest value of n is 25. That is, 25! is the smallest factorial that ends in exactly six zeros. (It could be noted that 26!, 27!, 28!, and 29! also end in six zeros.)
For the curious,

$$24! = \quad 620\ \ 448\ \ 401\ \ 733\ \ 239\ \ 439\ \ 360\ \ 000$$

and

$$25! = 15\ \ 511\ \ 210\ \ 043\ \ 330\ \ 985\ \ 984\ \ 000\ \ 000.$$

The number 24! ends in 4 zeros and the number 25! ends in 6 zeros.