$1 = 1^2$

$1 + 3 = 4 = 2^2$

$1 + 3 + 5 = 9 = 3^2$

$1 + 3 + 5 + 7 = 16 = 4^2$

Problem of the Week

Problem C and Solution

A Formula for That

Problem

When adding the first a positive odd integers to the first b positive odd integers, the sum is 180. If p is the largest odd number in the first set of numbers and q is the largest odd number in the second set of numbers, then determine the sum $p + q$.

Solution

Since there are a positive odd integers, we know that their sum is $a \times a = a^2$.
Since there are b positive odd integers, we know that their sum is $b \times b = b^2$.
We also know that when these two sets of odd numbers are added together, the sum is 180 so

$$a^2 + b^2 = 180.$$

We will proceed by selecting positive values for a, calculating a^2, and then determine if the remaining number required to sum to 180 is a perfect square. That is, if it is an integer that can be expressed as the product of two equal integers. The results are summarized in the table below.

a	a^2	$b^2 = 180 - a^2$	b $(b > 0)$	Solution?
1	1	$180 - 1 = 179$	13.4	no
2	4	$180 - 4 = 176$	13.3	no
3	9	$180 - 9 = 171$	13.1	no
4	16	$180 - 16 = 164$	12.8	no
5	25	$180 - 25 = 155$	12.4	no
6	36	$180 - 36 = 144$	12	yes
7	49	$180 - 49 = 131$	11.4	no
8	64	$180 - 64 = 116$	10.8	no
9	81	$180 - 81 = 99$	9.9	no
10	100	$180 - 100 = 80$	8.9	no
11	121	$180 - 121 = 59$	7.7	no
12	144	$180 - 144 = 36$	6	yes
13	169	$180 - 169 = 11$	3.3	no

If $a = 14$, then $a^2 = 196$. This produces a value greater than 180 and cannot be a possible solution.

It appears that there are two possible solutions.

When $a = 6$ and $b = 12$, $a^2 + b^2 = 36 + 144 = 180$. So adding the first 6 odd positive integers to the first 12 odd positive integers produces a sum of 180. The sixth odd positive integer is 11, so $p = 11$. The twelfth odd positive integer is 23, so $q = 23$. The sum, $p + q$, is then $11 + 23$ or 34. The second solution, $a = 12$ and $b = 6$, produces $p = 23$ and $q = 11$. The sum, $p + q$, is still 34.