Problem of the Month

Problem 0: September 2019

A point (x, y) in the plane is called a *lattice point* if it has integer coordinates. Points P, Q, and R are distinct lattice points. Prove that the measure of $\angle PQR$ cannot be 60°.

Problem of the Month

Problem 1: October 2019

Let $p(x)$ be the polynomial

$$\left(1+x\right)\left(1+x^3\right)\left(1+x^9\right)\left(1+x^{27}\right)\left(1+x^{81}\right)\cdots\left(1+x^{3^{20}}\right).$$

That is, $p(x)$ is the product of all binomials of the form $1+x^{3^k}$ where k ranges over the integers from 0 to 20 inclusive.

For each n, let c_n be the coefficient of x^n in $p(x)$.

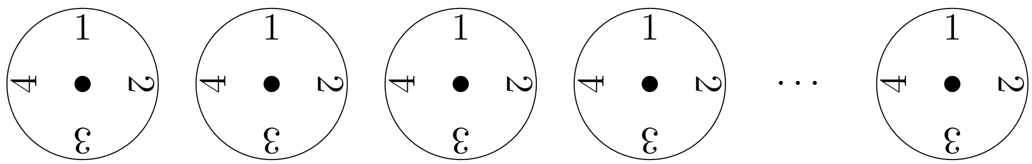
Compute the following sum of 1 000 001 coefficients of $p(x)$:

$$c_{1000000}+c_{1000001}+c_{1000002}+c_{1000003}+\cdots+c_{1999999}+c_{2000000}$$

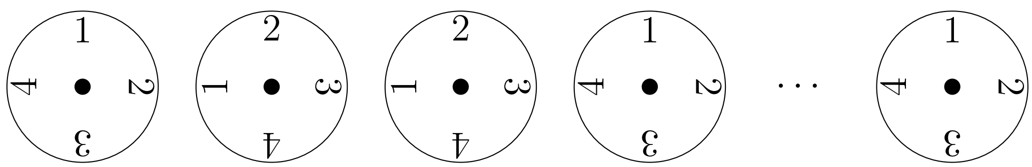
Problem of the Month

Problem 2: November 2019

(a) Several dials are each labelled with the integers from 1 through 4 in clockwise order. The dials are arranged in a row and initially configured so that each dial shows 1 at the top:



A *move* consists of rotating two adjacent dials in the same direction by the same number of positions. For example, one possible move is to rotate the second and third dials by one position in the counterclockwise direction resulting in the configuration shown:



There are $k \geq 2$ dials and they are initially configured so that each shows 1 at the top. In terms of k, determine how many possible configurations of the dials are attainable by performing a sequence of moves.

(b) Suppose now that there is an integer $n \geq 2$ so that each of the $k \geq 2$ dials is labelled in clockwise order by the integers 1 through n beginning with 1 at the top. Find the number of configurations of the dials that are attainable by a sequence of moves. Your answer should be in terms of n and k. Again, a move consists of rotating two adjacent dials in the same direction by the same number of positions.

(c) Answer the question in part (b) with the following additional type of move allowed: rotate the leftmost and rightmost dials in the same direction by the same number of positions.

Problem of the Month

Problem 3: December 2019

Let a, b, c, and d be rational numbers and $f(x) = ax^3 + bx^2 + cx + d$. Suppose $f(n)$ is an integer whenever n is an integer and that

$$\frac{1}{3}n^3 - n - \frac{2}{3} \leq f(n) \leq \frac{1}{3}n^3 + n^2 + 2n + \frac{4}{3}$$

for every integer n with the possible exception of $n = -2$.

(a) Show that $a = \frac{1}{3}$.

(b) Find $f(10^{2019}) - f(10^{2019} - 1)$.

Problem of the Month

Problem 4: January 2020

In this problem, we will call a rectangular prism an *appealing prism* if its three edge lengths are all integers and the length of the diagonal of each face is an integer.

(a) Show that at least two of the three edge lengths of an appealing prism must be multiples of 3.

(b) Show that the volume of an appealing prism must be a multiple of 1584.

(c) Find an appealing prism with shortest edge length equal to 44.

More to think about: It turns out that 44 is the smallest edge length that occurs in an appealing prism. Can you think of a way you might prove this? Do you think there are appealing prisms with integer-length *space diagonal*? A space diagonal of a rectangular prism is a line segment connecting any two vertices that are not on a common face. Note: there are three space diagonals and they all have the same length.

Problem of the Month

Problem 5: February 2020

(a) Suppose $0 < \alpha < 1$ is a rational number. Prove that there are positive integers $n_1 < n_2 < n_3 < \cdots < n_k$ with the property that

$$\alpha = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \cdots + \frac{1}{n_k}.$$

For example, with $\alpha = \frac{2}{3}$, we have $\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$. To begin, you may want to try to express a few specific rational numbers this way. For example, try $\alpha = \frac{2}{7}$.

(b) Suppose $0 < \alpha < 1$ is a rational number and T is a positive integer. Prove that there are positive integers $n_1 < n_2 < n_3 < \cdots < n_k$ each of which is greater than T with the property that

$$\alpha = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \cdots + \frac{1}{n_k}.$$

For example, try writing $\frac{2}{7}$ as a sum of reciprocals of distinct positive integers all of which are greater than 5.

(c) Let α be a positive rational number. Prove that there are positive integers $n_1 < n_2 < n_3 < \cdots < n_k$ with the property that

$$\alpha = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \cdots + \frac{1}{n_k}.$$

(d) The infinite series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

is known as the *harmonic series*. It is famously true that this series does not have a sum. More precisely, for any positive real number M, there is a natural number n so that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} > M.$$

In other words, the sum eventually exceeds any given real number, so it cannot equal any number. Give a proof of this fact using part (c).

Problem of the Month

Problem 6: March 2020

A point $A(a, b)$ in chosen in the (x, y)-plane. For a point P that lies on the graph of $y = x^3$, let M be the midpoint of AP. As the point P varies, so does the midpoint M of line segment AP. In fact, as P varies, the point M traces out the graph of another cubic equation.

(a) Find an equation for the cubic traced out by M. The coefficients of this cubic should depend on a and b.

(b) Describe all points $A(a, b)$ in the plane for which the traced-out cubic intersects $y = x^3$ in at least one but at most two distinct points.

Problem of the Month

Problem 7: April 2020

Define a function f whose input and output are both lists of n nonnegative integers by

$$f(a_1, a_2, \ldots, a_n) = (|a_1 - a_2|, |a_2 - a_3|, \ldots, |a_{n-1} - a_n|, |a_n - a_1|)$$

where, as usual, $|x|$ represents the absolute value of x.
For example, $f(1, 2, 3, 4) = (|1 - 2|, |2 - 3|, |3 - 4|, |4 - 1|) = (1, 1, 1, 3)$ and $f(2, 3, 5) = (|2 - 3|, |3 - 5|, |5 - 2|) = (1, 2, 3)$.
We will denote by f^k the function that *iterates* the application of f a total of k times. For example,

$$f^4(1, 1, 1, 3) = f^3(0, 0, 2, 2) = f^2(0, 2, 0, 2) = f(2, 2, 2, 2) = (0, 0, 0, 0).$$

We will call the list $(a_1, a_2, \ldots, a_n)$ *smooth* if there is some m for which $f^m(a_1, \ldots, a_n)$ is the list of n zeros. That is, a list is smooth if some number of applications of f will result in the list of all zeros. For example, $(1, 1, 1, 3)$ is smooth since $f^4(1, 1, 1, 3) = (0, 0, 0, 0)$, as demonstrated above.

(a) Find lists of length 5 and 7 that are not smooth.

(b) Show that for all odd integers $n \geq 1$ there exists a list L of length n that is not smooth.

(c) How many smooth lists (a, b, c) are there with a, b, and c each no larger than 100?

(d) Suppose L is a list of length 4 consisting of only zeros and ones. Show that L is smooth.

(e) Show that all lists of length 4 are smooth.

Problem of the Month

Problem 8: May 2020

For this problem, the *radius* of a square will be the distance from its centre to any of its four vertices. A *lattice point* is a point (a, b) in the plane where a and b are both integers.

(a) A square of radius $\frac{\sqrt{6}}{2}$ is placed in a random orientation with its centre at the origin. Two possible ways the square could be placed are pictured below. In Figure 1, there are five lattice points inside the square. In Figure 2, there is only one lattice point inside the square.

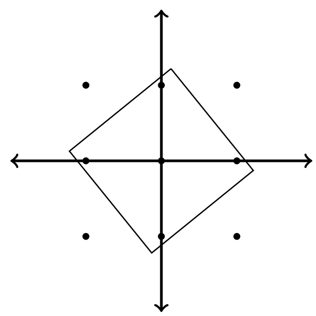


Figure 1

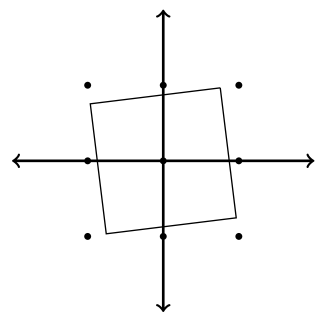


Figure 2

What is the probability that there are exactly 5 lattice points inside the square? For this and later parts of this problem, we will take the convention that a point on the perimeter of the square is inside the square.

Define a function f on the positive real numbers so that $f(r)$ is the largest possible number of lattice points inside any square of radius r centred at the origin.

(b) Show that $f\left(\frac{\sqrt{6}}{2}\right) = 5$.

(c) Show that $f(r)$ is always one more than a multiple of 4.

(d) Find the smallest positive real number r with the property that $f(r) = 17$.