Problem of the Month

Problem 8: May 2024

This month's problem is an extension of Question 4 from the 2024 Galois contest. It is not necessary to try the problem before attempting the questions below.

In an $m \times n$ rectangular grid, we say that two cells are *neighbours* if they share an edge. The *neighbourhood* of a cell is the cell itself along with its neighbours.

An $m \times n$ grid is called a *Griffin Grid* if each of its mn cells contains either a 1 or a -1 and the integer in every cell is equal to the product of the other integers in its neighbourhood.

For example, the 4×9 grid below is a Griffin Grid. The three shaded regions are the neighbourhoods of the cells in Row 1 and Column 1, Row 1 and Column 8, and Row 4 and Column 4.

-1	-1	1	1	1	1	1	-1	-1
1	1	-1	1	1	1	-1	1	1
-1	1	-1	-1	1	-1	-1	1	-1
-1	1	-1	1	1	1	-1	1	-1

The Galois problem restricted this definition to $m = 3$. Here we want to explore what happens more generally. If a question is marked with an asterisk ($*$), it means I was unable to solve it. Solutions will not be provided for these problems, but I would love to hear if you solve one!

(a) Show that an $m \times n$ grid with -1 or 1 in every cell is a Griffin Grid if and only if the cells in every neighbourhood have a product of 1.

(b) For every n, determine the number of $2 \times n$, $3 \times n$, and $4 \times n$ Griffin Grids. Determining the number of $3 \times n$ Griffin Grids in general is essentially what is required to answer part (c) of the Galois question.

(c) Show that the number of $m \times n$ Griffin Grids is of the form 2^k for some integer k with $0 \leq k \leq m$.

(d)* For general m, determine for which k there exists n with the property that the number of $m \times n$ Griffin Grids is exactly 2^k.

(e) Show that for all m there exist infinitely many n for which there is exactly one $m \times n$ Griffin Grid.

(f) Show that for all m there exist infinitely many n for which there are 2^m distinct $m \times n$ Griffin Grids.

(g)* Find a simple general way to calculate the number of $m \times n$ Griffin Grids.