Curves in the plane are often given as the set of points \((x, y)\) that satisfy some equation in \(x\) and \(y\). For example, the set of points \((x, y)\) that satisfy \(y = x^2\) is a parabola, the set of points \((x, y)\) that satisfy \(y = 3x + 4\) is a line, and the set of points \((x, y)\) that satisfy \(x^2 + y^2 = 1\) is the circle of radius 1 centred at the origin.

Another way to express a curve in the plane is using *parametric* equations. With this type of description, we introduce a third variable, \(t\), called the *parameter*, and each of \(x\) and \(y\) is given as a function of \(t\). This is useful for describing the position of a point that is moving around the plane. For example, imagine that an ant is crawling around the plane. If its \(x\)-coordinate at time \(t\) is \(x = x(t)\) and its \(y\)-coordinate at time \(t\) is \(y(t)\), then its position at time \(t\) is \((x(t), y(t))\).

(a) A particle’s position at time \(t\) is \((x, y) = (1 + t, −2 + 2t)\). That is, its \(x\)-coordinate at time \(t\) is \(1 + t\) and its \(y\)-coordinate at time \(t\) is \(−2 + 2t\).

(i) Plot the position of the particle at \(t = 0, 1, 2, 3,\) and 4.

(ii) Show that every position the particle occupies is on the line with equation \(y = 2x − 4\).

(iii) Sketch the path of the particle from \(t = 0\) through \(t = 4\).

(b) A particle’s position at time \(t\) is \((\cos(t), \sin(t))\). Sketch the path of the particle from \(t = 0\) through \(t = 2\pi\).

(c) A particle’s position at time \(t\) is \((\cos(t), \sin(2t))\).

(i) Plot the position of the particle at \(t = \frac{k\pi}{12}\) for the integers \(k = 0\) through \(k = 24\). Sketch the path of the particle from \(t = 0\) through \(t = 2\pi\).

(ii) Show that every position the particle occupies is on the curve with equation \(y^2 = 4x^2 − 4x^4\).

(d) Circle 1 is centred at the origin, Circle 2 is centred at \((2, 0)\), and both circles have radius 1. The circles are tangent at \((1, 0)\). Circle 2 is “rolled” in the counterclockwise direction along the outside of the circumference of Circle 1 without slipping. The point on Circle 2 that was originally at \((1, 0)\) (the point of tangency) follows a curve in the plane. Find functions \(x(t)\) and \(y(t)\) so that the points on this curve are \((x(t), y(t))\) for \(0 \leq t \leq 2\pi\).
(e) The setup in this problem is similar to (d). Circle 1 is centred at the origin and has radius 2 and Circle 2 is centred at (1, 0) and has radius 1 so that the two circles are tangent at (2, 0). Circle 2 is rolled around the inside of the circumference of Circle 1 in the counterclockwise direction. Describe the curve in the plane followed by the point on Circle 2 that is initially at (2, 0).

(f) Circle 1 is centred at the origin and has radius 1. Circle 2 has radius $r < 1$, is inside Circle 1, and the two circles are initially tangent at (1, 0). When Circle 2 is rolled around the inside of Circle 1 in the counterclockwise direction, the point on Circle 2 that was initially at (1, 0) will follow some curve in the plane.

(i) Show that when $r = \frac{1}{4}$, the points on the curve satisfy the equation $\left(\sqrt[3]{x}\right)^2 + \left(\sqrt[3]{y}\right)^2 = 1$.

(ii) Show that the curves for $r = \frac{1}{3}$ and $r = \frac{2}{3}$ are exactly the same and that the point initially at (1, 0) travels this curve in opposite directions for the two radii.