Problem of the Month
Problem 7: April 2024

Hint

(a) In part (ii), solve for \( t \) in terms of \( x \) and then substitute into the equation involving \( y \).

(b) At time \( t \), how far is the particle from the origin?

(c) Starting with \( y^2 = (\sin 2t)^2 \), use trigonometric identities to eliminate all appearances of the variable \( t \). Remember that \( x = \cos t \).

(d) As Circle 2 rolls around Circle 1, let \( t \) be the angle made by the positive \( x \)-axis and the line segment connecting the origin and the centre of Circle 2. It will help to draw a reasonably accurate diagram with Circle 2 rolled part of the way around Circle 1 (perhaps an angle of \( \frac{\pi}{3} \) or so). Once you have done this, mark the point on the circumference of Circle 2 that was originally at \((1,0)\) by \( P \) (or some other label). The objective is to find the coordinates of \( P \) in terms of \( t \). Since the circles roll without slipping, the arclength from the point of tangency along Circle 1 to \((1,0)\) should equal the arclength from the point of tangency along Circle 2 to \( P \).

(e) As Circle 2 rolls along the inside of Circle 1, it (usually) intersects the \( x \)-axis at two points. Convince yourself that one of these two points must be the origin, then think about the other point.

(f) Using a strategy similar to part (d), find a general formula for the coordinates of \( P \) in terms of the angle \( t \). Do this either in general for \( r \) or do it separately for the three relevant values of \( r \) in this question.

(i) Find identities for \( \cos 3t \) in terms of \( \cos t \) and \( \sin 3t \) in terms of \( \sin t \).

(ii) Find the parametric equations for the position of \( P \) when \( r = \frac{1}{3} \) if Circle 2 is rolled clockwise around Circle 1 instead of counterclockwise.