Problem of the Month

Problem 6: March 2024

This month's problem is an introduction to some of the basic ideas that come up when studying polynomials. It is presented with many exercises interspersed among some definitions. Some of the exercises are computational and some are theoretical.

Complex Numbers: The polynomial x^2+1 does not have any real roots. Because of this, we "invent" a root, and it is traditionally called i for "imaginary". That is, we declare that i is a number such that $i^2+1=0$ or $i^2=-1$. From a desire to do arithmetic with numbers "like" i, we declare that a *complex number* is a number of the form $a+bi$ where a and b are real numbers. For example, $1+i$ and $-3+2i$ are complex numbers. Remembering that $i^2=-1$ and using expected arithmetic rules, we can add and multiply complex numbers. For example,

$$\begin{aligned}(1+i)+(-3+2i) &= 1+i-3+2i = -2+3i \\ (1+i)(-3+2i) &= (1)(-3)+(1)(2i)+(i)(-3)+(i)(2i) \\ &= -3+2i-3i+2i^2 \\ &= -3+2i-3i+2(-1) \\ &= -5-i\end{aligned}$$

Using the quadratic formula, complex numbers give roots to all real quadratics. For example, the quadratic $f(x)=x^2-4x+13$ has a discriminant of $(-4)^2-4(13)=-36$, and so it has no real roots. However, if we accept that $\sqrt{-36}$ means "a number that squares to -36", we can infer that $6i$ or $-6i$ could reasonably be considered as $\sqrt{-36}$. Indeed, using the quadratic formula, we get

$$\frac{4\pm\sqrt{-36}}{2} = \frac{4\pm 6i}{2} = 2\pm 3i$$

If you are new to complex numbers, or you just want to have some fun, you might want to check that $f(2+3i)=0$ and $f(2-3i)=0$ using the methods for adding and multiplying complex numbers demonstrated above.

For the next three exercises, it will be useful to remember that if r is a root of a polynomial $p(x)$, then $x-r$ is a factor of $p(x)$.

(a) Find all roots of the polynomial $f(x)=x^4-2x^3-2x^2+8$.

(b) Find all roots of the polynomial $g(x)=x^4+2x^2+1$.

(c) Find all roots of the polynomial $h(x)=x^4-4$.

Definition: A polynomial $p(x)$ is called a *rational polynomial* if all of its coefficients are rational numbers. The set of rational polynomials is denoted by $\mathbb{Q}[x]$.

Definition: Suppose $p(x)\in\mathbb{Q}[x]$ has degree at least 1. We say that $p(x)$ is *reducible* in $\mathbb{Q}[x]$ (or *reducible over* $\mathbb{Q}$) if there are rational polynomials $u(x)$ and $v(x)$, both of degree at least 1, such that $p(x)=u(x)v(x)$. We say that $p(x)$ is *irreducible over* $\mathbb{Q}$ if it is not reducible over $\mathbb{Q}$. In other words, $p(x)$ is irreducible over $\mathbb{Q}$ if it cannot be factored as a product of rational polynomials of degree lower than that of $p(x)$.

(d) Determine whether each given polynomial is reducible or irreducible over $\mathbb{Q}$.

(i) $x^2 - 2$

(ii) $x^3 - 6x^2 + 11x - 6$

(iii) $x^3 + x + 1$

(iv) $2x - 5$

(v) $x^4 + 3x^2 + 2$

(vi) $x^4 + 1$

Definition: A complex number α is called *algebraic* if it is a root of some non-constant rational polynomial. That is, α is algebraic if there is a polynomial $p(x) \in \mathbb{Q}[x]$ of degree at least 1 such that $p(\alpha) = 0$. The *degree* of the algebraic number α is the smallest positive integer d such that there is a rational polynomial $p(x)$ of degree d with $p(\alpha) = 0$. [The word "positive" is important because every number α is a root of the constant 0 polynomial.]

(e) Show that $\sqrt{2}$, $1 + \sqrt[3]{2}$, $1 + 2i$, and $\sqrt{2} + \sqrt{3}$ are all algebraic numbers and find their degrees. It might be easier to find the degrees after thinking about some of the later questions.

(f) Suppose α is an algebraic number of degree d. Prove that there is a unique irreducible polynomial $m(x)$ of degree d with leading coefficient equal to 1 such that $m(\alpha) = 0$.

Note: Numbers that are not algebraic are called *transcendental* numbers. Two famous examples of transcendental numbers are π and e.

Definition: If $f(x)$ is a polynomial and α is a root of $f(x)$, then α is a *repeated root* of $f(x)$ if there is a polynomial $g(x)$ such that $f(x) = g(x)(x-\alpha)^2$. Note that α might be a complex number, which means $g(x)$ could have complex coefficients.

(g) Let $f(x) = x^7 - 3x^6 + 2x^5 - 2x^4 + x^3 + 5x^2 + 4$. Find complex numbers $r_1, r_2, \ldots, r_7$ such that $f(x) = (x - r_1)(x - r_2)\cdots(x - r_7)$. In other words, find all roots of $f(x)$.
Hint: Some of the roots are small integers and every root corresponds a linear factor.

(h) Given a polynomial $f(x)$ of degree n, if we expand the expression $f(x+y)$ (here, y is another variable), we can write $f(x + y)$ as

$$f(x + y) = f_0(x) + yf_1(x) + y^2f_2(x) + \cdots + y^nf_n(x)$$

for unique polynomials $f_0(x), f_1(x), \ldots, f_n(x)$.

(i) For the polynomial $f(x)$ from part (g), determine the polynomials $f_0(x)$ and $f_1(x)$ described above and find all roots of $f_0(x)$ and $f_1(x)$.

(ii) Suppose that $f(x)$ is a polynomial and that r is a root of $f(x)$. Prove that r is a repeated root of $f(x)$ if and only if r is a root of $f_1(x)$.

(j) Prove that if $p(x)$ and $q(x)$ are irreducible rational polynomials with a root in common, then there is a rational number c such that $p(x) = cq(x)$.

(k) Prove that an irreducible rational polynomial cannot have a repeated root.

Note: The *division algorithm for polynomials* might be useful in some of the later problems. It will be explained briefly in the hint.