Problem of the Month
Problem 3: December 2023

Hint

For positive integers m and n with $m \le n$, the integer mn is *not* in Row n if and only if there are integers a and b with $m < a \le b < n$ such that $mn = ab$. This fact will likely be useful in all parts of this problem.

(a) With the above fact in mind, convince yourself that if $n^2 - kn$ is not in Row n, then there must be positive integers x and y such that $x \le y < k$ and $n^2 - kn = (n-x)(n-y)$. Now solve for n and try to determine how to choose x and y to maximize this expression for n.

(b) If integers a, b, n, and p satisfy $mp = ab$ and p is prime, then at least one of a and b must be a multiple of p.

(c) The general formula is $f(pq) = (p-1)(q-1)$. In this part and the rest of the parts, you might find the following observation useful: If u and v are integers with $u < v$, then $u \le v - 1$.

(d) Consider the cases when d is even and when d is odd separately.

(e) To formulate a guess at how to find $f(n)$ in general, consider some values of n for which you know the value of $f(n)$ and list the positive factors of n in increasing order. If you are so inclined, you could write some computer code to compute $f(n)$ for some moderately sized values of n.

The value of $f(n)$ depends on how n factors, so it is probably unreasonable to expect a general algebraic expression for $f(n)$ similar to $f(pq) = (p-1)(q-1)$. Instead, try to find a simple procedure to compute $f(n)$ assuming that you already know all the positive factors of n.