Problem of the Month
Problem 0: September 2023

In this problem, $f$ will always be a function defined by $f(r) = \frac{ar + b}{cr + d}$ where $a$, $b$, $c$, and $d$ are integers. These integers will vary throughout the parts of the problem.

Given such a function $f$ and a rational number $r_1$, we can generate a sequence $r_1, r_2, r_3, \ldots$ by taking $r_n = f(r_{n-1})$ for each $n \geq 2$. That is, $r_2 = f(r_1)$, $r_3 = f(r_2)$, $r_4 = f(r_3)$, and so on. Unless there is some point in the sequence where $f(r_{n-1})$ is undefined, a sequence of this form can be made arbitrarily long.

These sequences behave in different ways depending on the function $f$ and the starting value $r_1$. This problem explores some those behaviours.

(a) Suppose $f(r) = \frac{2r - 1}{r + 2}$.

(i) With $r_1 = \frac{3}{2}$, compute $r_2$, $r_3$, and $r_4$.

(ii) Find a rational number $r_1$ with the property that $r_2$ is defined, but $r_3$ is not defined.

(b) Suppose $f(r) = \frac{r + 3}{2r - 1}$.

(i) With $r_1 = \frac{3}{7}$, compute $r_2$, $r_3$, $r_4$, and $r_5$.

(ii) Determine all rational values of $r_1$ with the property that there is some integer $n \geq 1$ for which $f(r_n)$ is undefined. For all other values of $r_1$, find simplified formulas for $r_{2023}$ and $r_{2024}$ in terms of $r_1$.

(c) Suppose $f(r) = \frac{r + 2}{r + 1}$.

(i) With $r_1 = 1$, compute $r_2$ through $r_9$. Write down decimal approximations of $r_2$ through $r_9$ (after computing them exactly).

(ii) Suppose $r$ is a positive rational number. Prove that

$$\left| \frac{f(r) - \sqrt{2}}{r - \sqrt{2}} \right| = \left| \frac{1 - \sqrt{2}}{r + 1} \right|$$

(iii) Suppose $r_1$ is a positive rational number. Prove that $|r_n - \sqrt{2}| < \frac{1}{2^{n-1}} |r_1 - \sqrt{2}|$ for each $n \geq 2$. Use this result to convince yourself that as $n$ gets large, $r_n$ gets close to $\sqrt{2}$, regardless of the choice of the positive value $r_1$. Can you modify $f$ slightly so that the sequence always approaches $\sqrt{3}$?

(d) Explore the behaviour of the sequences generated by various values of $r_1$ for each of the functions below. Detailed solutions will not be provided, but a brief discussion will.

$$f(r) = \frac{r - 3}{r - 2}, \quad f(r) = \frac{r - 1}{5r + 3}, \quad f(r) = \frac{r - 1}{r + 2}, \quad f(r) = \frac{2r + 2}{3r + 3}, \quad f(r) = \frac{r + 1}{r - 2}$$