Problem of the Month
Problem 4: January 2023

For each positive integer $k$, define a function $p_k(n) = 1^k + 2^k + 3^k + \cdots + n^k$ for each integer $n$. That is, $p_k(n)$ is the sum of the first $n$ perfect $k$th powers. It is well known that $p_1(n) = \frac{n(n+1)}{2}$.

(a) Fix a positive integer $n$. Let $S$ be the set of ordered triples $(a, b, c)$ of integers between 1 and $n+1$, inclusive, that also satisfy $a < c$ and $b < c$. Show that there are exactly $p_2(n)$ elements in the set $S$.

(b) With $S$ as in part (a), show that there are $\binom{n+1}{2} + 2\binom{n+1}{3}$ elements in $S$ and use this to show that

$$p_2(n) = \frac{n(n+1)(2n+1)}{6}$$

(c) For each $k$, show that there are constants $a_2, a_3, \ldots, a_k, a_{k+1}$ such that

$$p_k(n) = a_2 \binom{n+1}{2} + \cdots + a_k \binom{n+1}{k} + a_{k+1} \binom{n+1}{k+1}$$

for all $n$.

Note: Actually computing the constants gets more and more difficult as $k$ gets larger. While you might want to compute them for some small $k$, in this problem we only intend that you argue that the constants always exist, not that you deduce exactly what they are.

(d) Use part (c) to show that $p_3(n) = \frac{n^2(n+1)^2}{4}$ and $p_4(n) = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$.

(e) It follows from the fact in part (c) that $p_k(n)$ is a polynomial of degree $k+1$. With $k = 5$, this means there are constants $c_0, c_1, c_2, c_3, c_4, c_5$, and $c_6$ such that

$$p_5(n) = c_0 + c_1 n + c_2 n^2 + c_3 n^3 + c_4 n^4 + c_5 n^5 + c_6 n^6$$

Use the fact that $p_5(1) = 1$ and $p_5(n) - p_5(n-1) = n^5$ for all $n \geq 2$ to determine $c_0$ through $c_6$, and hence, derive a formula for $p_5(n)$.

(f) Show that $n(n+1)$ is a factor of $p_k(n)$ for every positive integer $k$ and that $2n+1$ is a factor of $p_k(n)$ for every even positive integer $k$. 