Problem of the Month

Solution to Problem 8: May 2022

In each part, the solid will be a figure that has rotational symmetry about the x-axis. For each x-value, this means if we slice the solid by a plane perpendicular to the x-axis at that x-value, the cross section of the solid will be a circle with its centre on the x-axis. Thus, to describe the solid of revolution in each part, we need to determine, for each x-value, the radius of this cross sectional circle. To do this, we will examine the corresponding cross sections of the cube. The radius of the circular cross section of the solid of revolution will be the distance from the x-axis to the point in the cross section of the cube that is farthest from the x-axis. The GeoGebra applets provided in the hint may be useful for visualizing these cross sections.

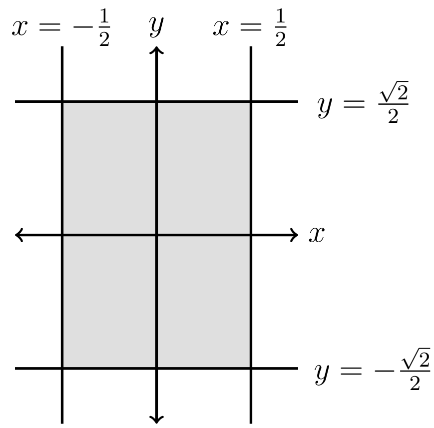
The approach in each part will be as follows:

- Determine the range of x-values occupied by the cube, which will be called I.
- For each $a \in I$, describe the cross section of the cube when it is sliced by the plane with equation $x = a$. [That is, the plane perpendicular to the x-axis that intersects the x-axis at $x = a$.]
- Let f be the function with domain I so that for each $a \in I$, $f(a)$ is the largest possible distance to the x-axis from a point in the cross section of the cube at $x = a$.
- The solid of revolution is that which has circular cross sections with radius $f(a)$ at each $a \in I$.
- The region in the xy-plane is the set of points with $x \in I$ that are above the graph of $y = -f(x)$ and below the graph of $y = f(x)$.

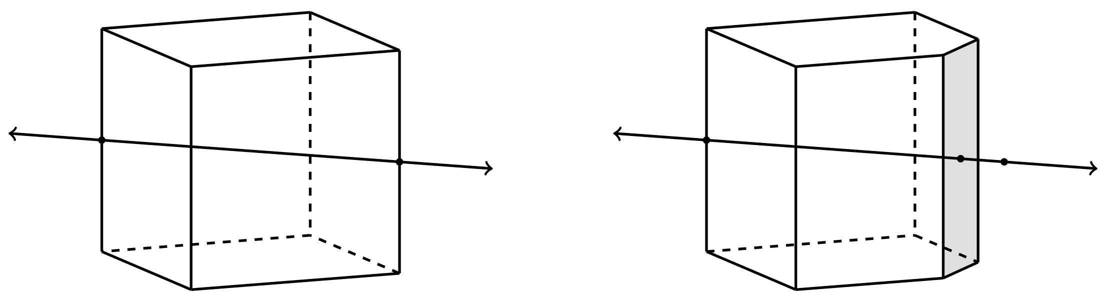
(a) Since the cube is centred at the origin and its sides have length 1, the cube is initially positioned along the interval $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$. Because the x-axis is perpendicular to two faces of the cube, when we slice the cube by any plane that is perpendicular to the x-axis, the cross section is a unit square with its centre on the x-axis.

In any square, the points that are farthest from the centre are the four vertices. The distance from the centre to a vertex is half the length of the diagonal of the square. By the Pythagorean theorem, the length of the diagonal of a unit square is $\sqrt{1^2 + 1^2} = \sqrt{2}$, so the distance from the centre of the square to a vertex is $\frac{\sqrt{2}}{2}$.

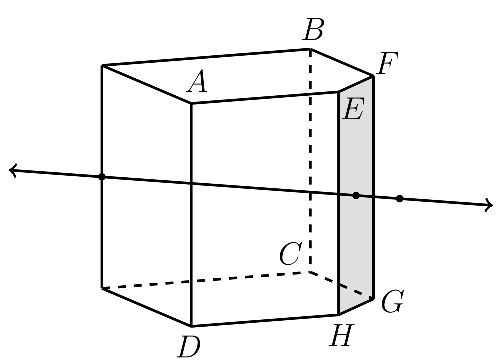
Thus, for any $a \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, we have that $f(a) = \frac{\sqrt{2}}{2}$. The solid of revolution is the cylinder that is parallel to the x-axis that has radius $\frac{\sqrt{2}}{2}$ and height 1. The cylinder intersects the xy-plane in a rectangle, and that rectangle is the set of points that are bound by the horizontal lines with equations $y = \frac{\sqrt{2}}{2}$ and $y = -\frac{\sqrt{2}}{2}$ on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. This region is pictured below.



(b) The distance between the midpoints of two opposite edges of a cube is the same as the length of the diagonal of any face. By the computation in part (a), this length is $\sqrt{2}$. Since the cube is centred at the origin, the interval I in this part is $\left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$. In the diagram below, the cube is seen in its original position. On the left, the entire cube is pictured. On the right, a piece has been removed to show a generic cross section, which is shaded.



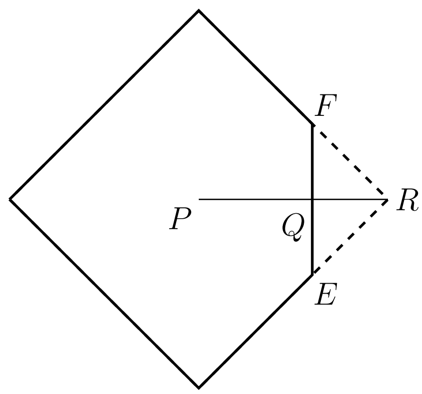
The shaded cross section appears to be a rectangle. To explain why it is indeed a rectangle, we first note that the plane with equation $x = a$ with $a > 0$ intersects four faces of the cube, and each of these intersections gives a line segment. This means that the cross section is a quadrilateral. To see that this quadrilateral is indeed a rectangle, we will label some points on the surface of the cube. The vertices of the cube that are on the plane with equation $x = 0$ will be labelled A, B, C, and D with A at the "top front", B at the "top back", C at the "bottom back", and D at the "bottom front". As well, the points where the plane with equation $x = a$ intersect the edges of the cube will be labelled E, F, G, and H in such a way that segments AE, BF, CG, and DH all lie on edges of the cube.



The plane through A, B, C, and D is perpendicular to the x-axis because of the way the cube is positioned. Therefore, the plane through A, B, C, and D is parallel to the plane with equation $x = a$. Hence, AD, BC, FG, and EH are all parallel. By similar reasoning, EF is parallel to HG, so $EFGH$ is a parallelogram. As well, AD is perpendicular to the top face of the cube, which means EH is perpendicular to the top face of the cube. This means that EH is perpendicular to any line through E and another point in the top face. Hence, EH is perpendicular to EF. A parallelogram with a right angle must be a rectangle, which shows that $EFGH$ is a rectangle. Also note that $AEHD$ is a rectangle for similar reasoning.

By symmetry, the centre of the rectangular cross section is on the x-axis. Thus, the circular cross section at $x = a$ of the solid of revolution has a radius equal to the distance from the centre of the rectangular cross section of the cube to any of its four vertices. This radius is half the length of the diagonal of the rectangular cross section, which can be found using the Pythagorean theorem once we know the side lengths, so it remains to determine the dimensions of the cross section at $x = a$, which we expect to depend on the value of a. By symmetry, it is enough to consider $a > 0$.

As noted above, $AEHD$ is a rectangle, so $EH = AD = 1$, which is independent of the value of a. The length of EF does depend on the value of a. Below is a diagram of the top face of the cube. The centre of the top face has been labelled by P, the corner of the top face that was removed in the previous diagram is labelled by R, and the point where the line segment PR intersects EF is labelled by Q.



The length of PR is equal to $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$ because it is half the length of the diagonal of a unit square. As well, PQ has length a by assumption, and EF is perpendicular to PQ because PQ is parallel to the x-axis and EF is perpendicular to the x-axis. Therefore, QR is an altitude of $\triangle FQR$. The line connecting the centre of a square to one of its vertices must be an angle bisector, which means that $\angle PRF = \angle PRE = 45^\circ$. Since $\angle FQR = \angle EQR = 90^\circ$, we also have $\angle QFR = \angle QER = 45^\circ$, which means that $\triangle FQR$ and $\triangle EQR$ are both isosceles. Therefore, $QE = QR = QF$, but $QR = \frac{1}{\sqrt{2}} - a$, so

$$EF = QE + QF = 2QR = 2\left(\frac{1}{\sqrt{2}} - a\right) = \sqrt{2} - 2a$$

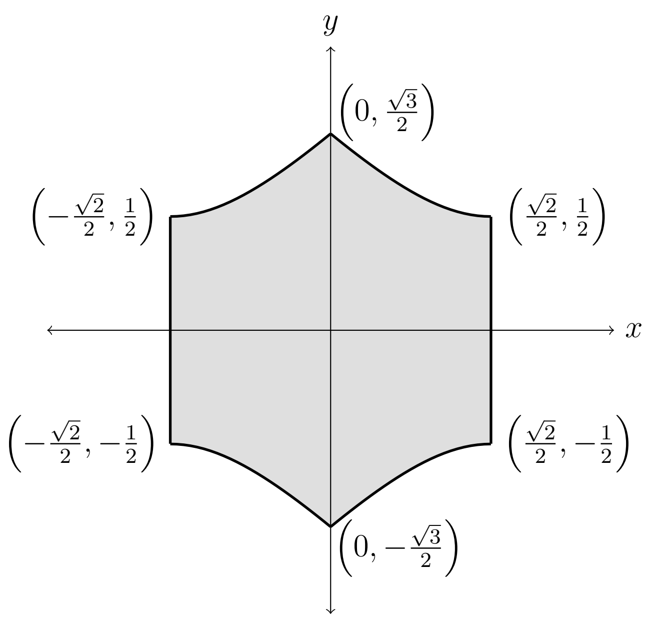
Therefore, the diagonal length of the rectangular cross section at $x = a$ is

$$\sqrt{EH^2 + EF^2} = \sqrt{1^2 + (\sqrt{2} - 2a)^2} = \sqrt{3 - 4\sqrt{2}a + 4a^2}$$

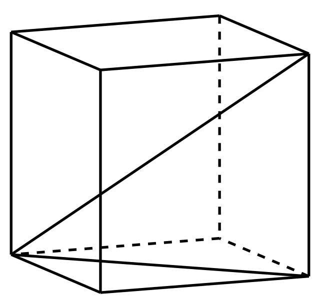
For $x \geq 0$, we have that $f(x) = \frac{1}{2}\sqrt{3 - 4\sqrt{2}x + 4x^2}$. By symmetry, the function f on the interval should be an *even* function on I, which means $f(x) = f(-x)$. This means that we can replace x by $-x$ to determine $f(x)$ when $x < 0$. After doing this, we find that $f(x)$ is defined *piecewise* by

$$f(x) = \begin{cases} \frac{1}{2}\sqrt{3 + 4\sqrt{2}x + 4x^2} & \text{if } -\frac{\sqrt{2}}{2} \leq x < 0 \\ \frac{1}{2}\sqrt{3 - 4\sqrt{2}x + 4x^2} & \text{if } 0 \leq x \leq \frac{\sqrt{2}}{2} \end{cases}$$

Below is a diagram of the region above the graph of $y = -f(x)$ and below the graph of $y = f(x)$ on the interval $\left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$.



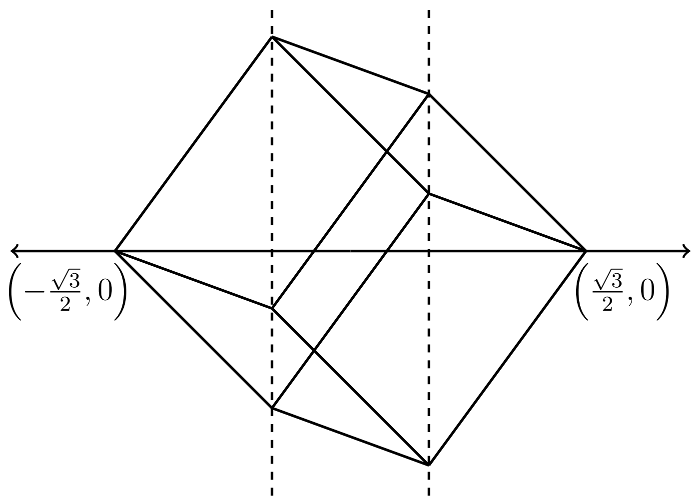
(c) In this part, we observe that the distance between two opposite vertices of a cube is the length of the hypotenuse of a right-angled triangle with one leg equal to an edge of the cube and one leg equal to the diagonal of a face of the cube. This is pictured below.



The length of the diagonal of a unit square is $\sqrt{2}$, so the distance between two opposite vertices of the cube is $\sqrt{(\sqrt{2})^2 + 1^2} = \sqrt{3}$. Thus, in this part, the interval is $I = \left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$.

In this part, the cross sections come in two "types". If x is close enough to 0, then the cross section is a hexagon. Otherwise, the cross section is an equilateral triangle.

Consider the vertices of the cube that are to the right of the origin. By rotational symmetry, if the cube is rotated $120°$ around the x-axis, these vertices (other than those on the x-axis) will take each other's positions. Therefore, they must all have the same x-coordinate, so there is some $\alpha > 0$ such that the plane with equation $x = \alpha$ passes through all three of these vertices. Similarly, there is $\beta < 0$ so that the plane with equation $x = \beta$ passes through all three of the vertices of the cube that are not on the x-axis and have a negative x-coordinate. The diagram below is of a cube positioned with two opposite vertices on the x-axis but viewed at an angle perpendicular to the x-axis. This gives some indication of the different cross sections and where they change type. The dashed vertical lines are meant to represent the planes with equations $x = \alpha$ and $x = \beta$.



Suppose $\alpha < a < \frac{\sqrt{3}}{2}$. The plane with equation $x = a$ intersects three faces of the cube, so the cross section of the cube at $x = a$ is a triangle. The rotational symmetry of the cube implies that this triangle has $120°$ rotational symmetry about the x-axis, and such a triangle must be equilateral since it must have three equal angles. Similarly, if $-\frac{\sqrt{3}}{2} < a < \beta$, then the cross section at $x = a$ is also an equilateral triangle.

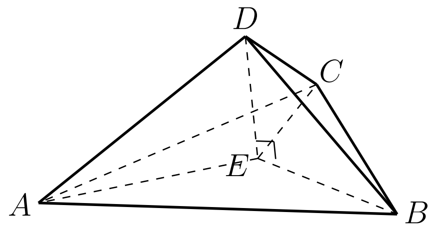
For $\beta < a < \alpha$, the plane with equation $x = a$ intersects all 6 faces of the cube. The plane intersects each face in a line segment, so the cross section must be a hexagon since each of these line segments will be a side of the cross section. There is no reason to expect it to be a regular hexagon, but it will have $120°$ rotational symmetry, which will be used later.

We will delay computing the values of α and β, though we will observe that, by symmetry, $\alpha = -\beta$, and it suffices to analyze the cross sections at $x = a$ for $a > 0$.

To analyze the triangular cross sections, we will use the following fact.

Fact 1: Suppose tetrahedron $ABCD$ has equilateral base $\triangle ABC$ and its other three faces satisfy $\angle ADB = \angle BDC = \angle CDA = 90°$ and $AD = BD = CD$. If E is the point in $\triangle ABC$ so that DE is the altitude of the tetrahedron from D, then $DE = \frac{AD}{\sqrt{3}}$ and $AE = \sqrt{2}DE$.

Proof. By symmetry, $AE = BE = CE$.



Since $\triangle ADB$ is isosceles and right-angled with hypotenuse AB, we get that $AB = \sqrt{2}AD$. We also have that $\triangle AEB$, $\triangle BEC$, and $\triangle CEA$ are all congruent by side-side-side congruence. As well, $\angle AEB + \angle BEC + \angle CEA = 360°$, so since they are equal by congruence, they are all equal to $120°$. Because $AE = BE$, it follows that $\triangle AEB$ is isosceles and that $\angle ABE = \dfrac{180° - 120°}{2} = 30°$.

Using the Sine law, we have $\dfrac{AE}{\sin 30°} = \dfrac{AB}{\sin 120°}$, from which it follows that

$$AE = \frac{AB \sin 30°}{\sin 120°} = \frac{AB}{\sqrt{3}} = \frac{\sqrt{2}AD}{\sqrt{3}}$$

We can now use the Pythagorean theorem on $\triangle ADE$ to get that

$$DE = \sqrt{AD^2 - AE^2} = \sqrt{AD^2 - \left(\frac{\sqrt{2}}{\sqrt{3}}AD\right)^2} = AD\sqrt{1 - \frac{2}{3}} = \frac{AD}{\sqrt{3}}$$

which is one of the claims in the fact. The other now follows by rearranging the equation above to get $AD = \sqrt{3}DE$ then substituting into $AE = \dfrac{\sqrt{2}AD}{\sqrt{3}}$ to get

$$AE = \frac{\sqrt{2}AD}{\sqrt{3}} = \frac{\sqrt{2}(\sqrt{3}DE)}{\sqrt{3}} = \sqrt{2}DE$$

□

We can now compute the value of α as well as $f(a)$ for each a with $\alpha < a < \dfrac{\sqrt{3}}{2}$.

When we take the cross section at $x = a$ with $\alpha \leq a < \dfrac{\sqrt{3}}{2}$, we have already argued that the cross section is an equilateral triangle. Taking such a cross section "removes" a tetrahedral corner of the cube with this cross section as its base. By rotational symmetry and the fact that the faces of a cube are squares, the other three faces of this "removed" tetrahedron are isosceles right-angled triangles. Thus, the tetrahedron satisfies the conditions of Fact 1. Moreover, points E and D are on the x-axis and $f(a)$ is equal to the length of AE.

When $a = \alpha$, AD is an edge of the cube, so $AD = 1$ which gives $DE = \dfrac{AD}{\sqrt{3}} = \dfrac{1}{\sqrt{3}}$ by Fact 1. As well, $\alpha = \dfrac{\sqrt{3}}{2} - DE$, which means

$$\alpha = \frac{\sqrt{3}}{2} - DE = \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

For any a with $\frac{1}{2\sqrt{3}} \le a < \frac{\sqrt{3}}{2}$, the tetrahedron has $DE = \frac{\sqrt{3}}{2} - a$, and since $AE = \sqrt{2}DE$, we get

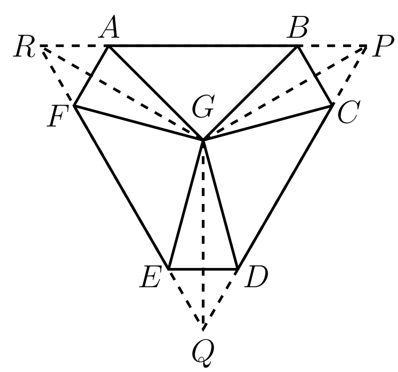
$$f(a) = AE = \sqrt{2}DE = \sqrt{2}\left(\frac{\sqrt{3}}{2} - a\right) = \frac{\sqrt{3}}{\sqrt{2}} - \sqrt{2}a$$

While we have not yet determined how to compute $f(x)$ for all $x \in I$, we do now have for $x \in \left[\frac{1}{2\sqrt{3}}, \frac{\sqrt{3}}{2}\right)$ that $f(x) = \frac{\sqrt{3}}{\sqrt{2}} - \sqrt{2}x$. Notice that at $x = \frac{\sqrt{3}}{2}$, $f(x) = 0$, which makes sense. You may want to think about this.

Next we will examine the hexagonal cross sections for $0 \le a < \frac{1}{2\sqrt{3}}$. We will use the following fact.

<u>Fact 2</u>: Suppose that $ABCDEF$ is a hexagon that has opposite sides parallel (that is, AB and DE are parallel, BC and EF are parallel, and CD and FA are parallel) and has a point G in its interior so that the hexagon has 120° rotational symmetry about G. Then G is equidistant from all six vertices of the hexagon.

Proof. Below is a diagram of such a hexagon with AB and DC extended to meet at P, CD and FE extended to meet at Q, and EF and BA extended to meet at R. Point G is also connected to each vertex of the hexagon as well as to P, Q, and R.



The fact that the hexagon has 120° rotational symmetry means that it also has 240° rotational symmetry. This means that it has 120° rotational symmetry both clockwise and counterclockwise. A clockwise rotation will send A to the position of C, B to D, C to E, D to F, E to A, and F to B. Since the rotation is around G, this implies that $GA = GC = GE$ and $GB = GD = GF$, as well as $AB = CD = EF$ and $BC = DE = FA$. Finally, by the way P, Q, and R are defined, the rotational symmetry also implies that $\triangle PQR$ has 120° rotational symmetry about G. From an earlier argument, this implies $\triangle PQR$ is equilateral and that G is equidistant from P, Q, and R.

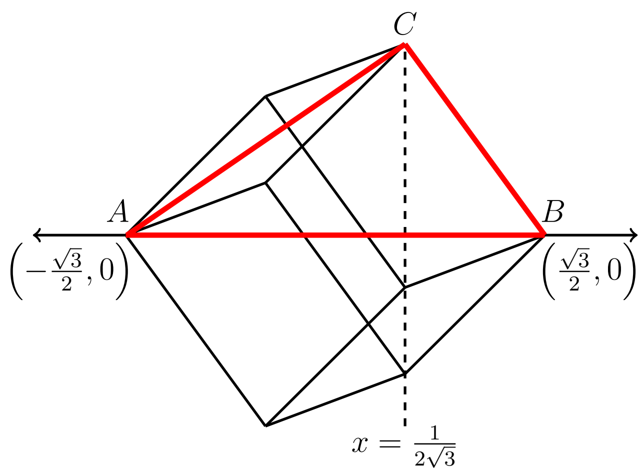
By properties of parallel lines, we get that $\angle RAF = \angle EDQ = \angle BPC$, but from the previous paragraph we have $\angle BPC = \angle ARF = 60°$, so $\triangle RAF$ has two angles equal to 60°. Therefore, it is equilateral.

Since $GP = GR = GQ$ and $PQ = QR = RP$, it must be that $\triangle GPQ$, $\triangle GQR$, and $\triangle GRP$ are all congruent by side-side-side congruence. It follows that $\angle GRQ = \angle GRP$, and since their sum is $60°$, they are both equal to $30°$. Let H be the point of intersection of GR and AF. We know that $\angle RAF = 60°$, and so it follows that $\angle RHA = 90°$. As well, $\triangle FRH$ and $\triangle ARH$ are congruent by side-angle-side congruence, so $FH = AH$. Since $\angle RHA = 90°$, so do each of $\angle FHG$ and $\angle AHG$, so we now conclude that $\triangle FHG$ and $\triangle AHG$ are congruent by side-angle-side congruence. This means $GA = GF$, and since $GA = GC = GE$ and $GB = GD = GF$, it follows that G is equidistant from all six vertices of the hexagon. $\square$

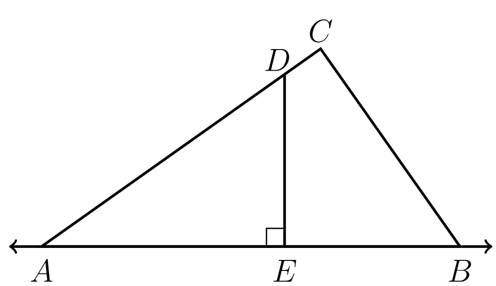
The cube has $120°$ rotational symmetry about the x-axis, and so if we take any a with $0 \le a < \frac{1}{2\sqrt{3}}$, the hexagonal cross section must also have $120°$ rotational symmetry about the point where the plane with equation $x = a$ intersects the x-axis. As well, opposite faces of the cube are parallel, so opposite sides of the hexagonal cross section must also be parallel. By Fact 2, the six vertices of the cross section are equidistant from the x-axis.

This means that $f(a)$ is the distance from the x-axis to any of the six points where the plane with equation $x = a$ intersects an edge of the cube.

Consider the diagram of the cube below. The vertices of the cube that are on the x-axis are labelled by A and B, one of the vertices of the cube with x-coordinate equal to $\frac{1}{2\sqrt{3}}$ is labelled by C, and $\triangle ABC$ is in bold red.



We know that $AB = \sqrt{3}$, $AC = \sqrt{2}$ and $BC = 1$. Suppose $0 \le a < \frac{1}{2\sqrt{3}}$ and consider the cross section at $x = a$. Let E be the point on the x-axis at $x = a$, which implies that E is on AB. As well, let D be the point at which the plane with equation $x = a$ intersects AC. The plane is perpendicular to the x-axis, which means that $\angle AED = 90°$.



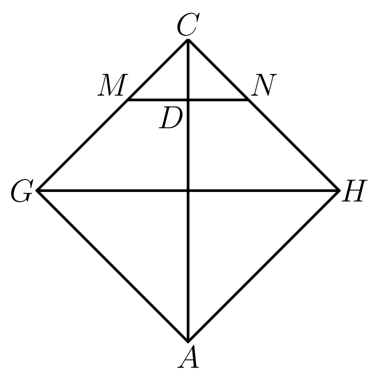
The length of AE is $\dfrac{\sqrt{3}}{2}+a$, and $\triangle AED$ is similar to $\triangle ACB$ since they share an angle at A and $\angle AED = \angle ACB = 90^\circ$. Therefore, $\dfrac{AD}{AE} = \dfrac{AB}{AC} = \dfrac{\sqrt{3}}{\sqrt{2}}$ which gives

$$AD = \frac{\sqrt{3}}{\sqrt{2}}\left(\frac{\sqrt{3}}{2}+a\right) = \frac{3}{2\sqrt{2}} + \frac{\sqrt{3}a}{\sqrt{2}}$$

As well, $\dfrac{ED}{AE} = \dfrac{CB}{AC} = \dfrac{1}{\sqrt{2}}$ and so

$$ED = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2}+a\right) = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{a}{\sqrt{2}}$$

Next, let G and H be the other two vertices of the cube that are on the same face as A and C and let M and N be the points where the plane with equation $x = a$ intersects GC and HC, respectively.



The plane with equation $x = -\dfrac{1}{2\sqrt{3}}$ contains both G and H and is parallel to the plane with equation $x = a$. Since segments GH and MN are themselves in the same plane, they must be parallel. It follows that $\triangle CMN$ is an isosceles right-angled triangle. By an argument used in part (b), it follows that $MD = CD = ND$. We have that $AC = \sqrt{2}$ and $AD = \dfrac{3}{2\sqrt{2}} + \dfrac{\sqrt{3}a}{\sqrt{2}}$, which means

$$MD = ND = CD = AC - AD = \sqrt{2} - \left(\frac{3}{2\sqrt{2}} + \frac{\sqrt{3}a}{\sqrt{2}}\right) = \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}a}{\sqrt{2}}$$

Now consider $\triangle EMN$ with D on MN. By the fact from earlier about hexagons, we already know that $EM = EN$. We have just shown that $MD = ND$. Since they also share side ED, we get that $\triangle EDN$ and $\triangle EDM$ are congruent by side-side-side congruence. Thus, $\angle EDM = \angle EDN$ and their sum is 180°, so $\triangle EDN$ is right-angled at D. Therefore, the length of EN, which is $f(a)$, can be computed using the Pythagorean theorem.

$$\begin{aligned}
f(a) = EN &= \sqrt{ED^2 + ND^2} \\
&= \sqrt{\left(\frac{\sqrt{3}}{2\sqrt{2}} + \frac{a}{\sqrt{2}}\right)^2 + \left(\frac{1}{2\sqrt{2}} - \frac{\sqrt{3}a}{\sqrt{2}}\right)^2} \\
&= \sqrt{\frac{3}{8} + \frac{\sqrt{3}a}{2} + \frac{a^2}{2} + \frac{1}{8} - \frac{\sqrt{3}a}{2} + \frac{3a^2}{2}} \\
&= \sqrt{\frac{1}{2} + 2a^2}
\end{aligned}$$

We can now define $f(x)$ on $\left[0, \frac{\sqrt{3}}{2}\right]$ as a piecewise function:

$$f(x) = \begin{cases} \sqrt{\frac{1}{2} + 2x^2} & \text{if } 0 \le x < \frac{1}{2\sqrt{3}} \\ \frac{\sqrt{3}}{\sqrt{2}} - \sqrt{2}x & \text{if } \frac{1}{2\sqrt{3}} \le x \le \frac{\sqrt{3}}{2} \end{cases}$$

To extend f to all of I, we observe that, like in part (b), $f(x) = f(-x)$. Thus, we can define $f(x)$ on $\left[-\frac{\sqrt{3}}{2}, 0\right]$ by substituting $x = -x$. Note that since $(-x)^2 = x^2$, the function definition is the same for $-\frac{1}{2\sqrt{3}} < x \le 0$ as it is for $0 \le x < \frac{1}{2\sqrt{3}}$. Thus, we get that

$$f(x) = \begin{cases} \frac{\sqrt{3}}{\sqrt{2}} + \sqrt{2}x & \text{if } -\frac{\sqrt{3}}{2} \le x \le -\frac{1}{2\sqrt{3}} \\ \sqrt{\frac{1}{2} + 2x^2} & \text{if } -\frac{1}{2\sqrt{3}} < x < \frac{1}{2\sqrt{3}} \\ \frac{\sqrt{3}}{\sqrt{2}} - \sqrt{2}x & \text{if } \frac{1}{2\sqrt{3}} \le x \le \frac{\sqrt{3}}{2} \end{cases}$$

Below is a diagram of the region above the graph of $y = -f(x)$ and below that of $y = f(x)$.