Problem of the Month

Problem 8: May 2022

Problem

In each part of this problem, a unit cube is positioned with its centre at the origin and is rotated about the x-axis so that it sweeps out a new "solid of revolution". To visualize this solid, you might imagine a cube being rotated very quickly about a fixed axis to produce an illusion of the solid. [This is the same phenomenon as when a rotating propellor or fan blade looks like a disk.] For example, if the cube is originally positioned so that the x-axis passes through the centres of two opposite faces, then the solid of revolution is a cylinder.

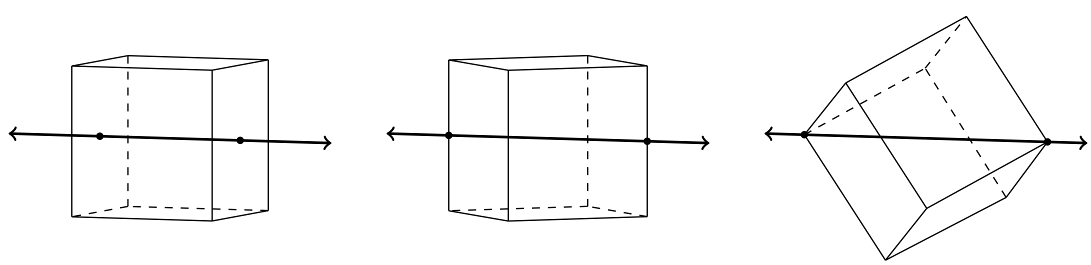
The solid of revolution depends on the original position of the cube. In each part, information is given to describe the original position of the cube and the goal is to describe the region in the (x, y)-plane intersected by the solid of revolution.

(a) The cube is positioned so that the x-axis passes through the centres of two opposite faces. As mentioned in the preamble, the solid is a cylinder.

(b) The cube is positioned so that the x-axis passes through the midpoints of two opposite edges of the cube (that is, two edges that are parallel and are not edges of the same face).

(c) The cube is positioned so that the x-axis passes through two opposite vertices of the cube (that is, two vertices that are not on a common face).

Below, from left to right, are diagrams of the original position of the cube for parts (a), (b), and (c), respectively. In order to avoid clutter in the diagrams, only the x-axis is included.



Notes:

- In the solutions, regions in the (x, y)-plane will have descriptions of the form "the region between $x = a$ and $x = b$ above the graph of $y = f(x)$ and below the graph of $y = g(x)$. You may have some other way of describing the regions.
- Solids of revolution are studied in calculus. If you already know some calculus and would like an added challenge, you might like to try to compute the area of the regions you find, or even the volumes of the solids of revolution.

Hint

(a) The cross sections of the cube in this part are all unit squares. What length in the square is equal to the diameter of the base of the cylinder swept out by the rotating the cube?

(b) In this part, the cross sections are always rectangles, but their dimensions vary depending on where the cross section is taken. Here is a link to a GeoGebra applet to help visualize the rotating cube and the cross sections. The "Rotate" option will cause the cube to rotate around the axis. The "Show Cross Section" option will show the cross section when the cube is sliced by a plane perpendicular to the axis. The plane can be moved to see different cross sections. The "Show Trace" option will show the solid traced out by the cube as it rotates.

(c) In this part, the cross sections are either triangles or hexagons, depending on how close the cross section is taken to the vertices that are fixed on the axis. The cube has 120° rotational symmetry, which means that if it is rotated by 120°, it occupies exactly the same space that it occupied before it was rotated. The cross sections must also have 120° rotational symmetry. It may be useful to think about what sorts of triangles and hexagons can have such symmetry. This is a link to another GeoGebra applet that works in essentially the same way as the one for part (b).