Problem of the Month

Solution to Problem 7: April 2022

(a) Assume that $abc = def$ and that $a + b + c = d + e + f$. To prove the claimed fact, we will further assume that $a = d = 1$ and deduce that either $b = e$ and $c = f$ or $b = f$ and $c = e$. This will prove that the 3-factorizations (a, b, c) and (d, e, f) are the same. Keep in mind that the integers in a 3-factorizations are positive.

With $a = d = 1$, the assumed equations become $bc = ef$ and $b+c = e+f$. Squaring both sides of the second equation leads to $b^2+2bc+c^2 = e^2+2ef+f^2$. Since $bc = ef$, $4bc = 4ef$, and if we subtract this from $b^2+2bc+c^2 = e^2+2ef+f^2$ we get $b^2-2bc+c^2 = e^2-2ef+f^2$ which factors as $(b-c)^2 = (e-f)^2$. Taking square roots, this means $|b-c| = |e-f|$, so either $b-c = e-f$ or $b-c = f-e$.

Suppose $b-c = e-f$. Adding $b+c = e+f$ gives $2b = 2e$ or $b = e$. Similarly, if $b-c = f-e$, then $2b = 2f$ or $b = f$. Thus, either $b = e$ or $b = f$. If $b = e$, then $b+c = e+f$ implies $c = f$. If $b = f$, then $b+c = e+f$ implies $c = e$. Therefore, either $b = e$ and $c = f$ or $b = f$ and $c = e$.

We have shown that if two 3-factorizations of an integer each contain a 1 and have the same sum, then they must be the same 3-factorization. Therefore, it is impossible for two different 3-factorizations of an integer to have the same sum and both contain 1.

(b) Suppose n is either prime or is the product of 2 prime numbers. Then any 3-factorization of n must contain a 1, so part (a) implies that it is impossible for two different 3-factorizations of n to have the same sum.

We now suppose $n = pqr$ where p, q, and r are prime, some or all of which may be equal. The only 3-factorization of n that does not include a 1 is (p, q, r), so if there are two 3-factorizations of n with the same sum, then part (a) implies that one of these 3-factorizations is (p, q, r).

The other 3-factorizations of n are $(1, 1, pqr)$, $(1, p, qr)$, $(1, q, pr)$, and $(1, r, pq)$. We will show that none of the equations

$$\begin{aligned}
p+q+r &= 1+1+pqr \\
p+q+r &= 1+p+qr \\
p+q+r &= 1+q+pr \\
p+q+r &= 1+r+pq
\end{aligned}$$

can be satisfied when p, q, and r are prime.

First, suppose $p+q+r = 1+1+pqr$ for some prime numbers p, q, and r. By possibly relabelling, we can assume that $r \geq q$ and $r \geq p$. Since p and q are both prime, $p \geq 2$ and $q \geq 2$, so $pq \geq 4$, which means $pqr \geq 4r$. Therefore, $1+1+4r \leq 1+1+pqr$, but we are assuming that $1+1+pqr = p+q+r$, so we have $1+1+4r \leq p+q+r$. Since $r \geq q$ and $r \geq p$, this means $2+4r \leq 3r$, but this is impossible since r is a positive integer. Therefore, it is impossible that $p+q+r = 1+1+pqr$ when p, q, and r are prime.

Now assume that $p+q+r=1+p+qr$ for some integers p, q, and r. This simplifies to $qr-q-r+1=0$ or $(q-1)(r-1)=0$, which means that either $q=1$ or $r=1$. Since 1 is not a prime number, the integers p, q, and r cannot all be prime, and so the equation cannot hold if p, q, and r are all prime.

By symmetry, $p+q+r=1+q+pr$ and $p+q+r=1+r+pq$ also cannot be satisfied when p, q, and r are all prime.

We have now shown that if n is prime, the product of two prime numbers, or the product of three prime numbers, then it cannot have two different 3-factorizations with the same sum. Therefore, if n has two different 3-factorizations with the same sum, then it must have at least four prime factors.

(c) By part (b), we can restrict our search to positive integers that are the product of at least four prime numbers. It can be checked that the first five positive integers with this property are 16, 24, 32, 36, and 40. In fact, 36 is the integer we seek, but we will go through the possibilities above in order to verify that 36 is indeed the smallest.

The 3-factorizations of 16 are $(1,1,16)$, $(1,2,8)$, $(1,4,4)$, and $(2,2,4)$. Their sums are 18, 11, 9, and 8, no two of which are the same, so 16 does not have two different 3-factorizations with the same sum.

The 3-factorizations of 24 are $(1,1,24)$, $(1,2,12)$, $(1,3,8)$, $(1,4,6)$, $(2,2,6)$, and $(2,3,4)$. Their sums are 26, 15, 12, 11, 10, and 9, no two of which are the same, so 24 does not have two different 3-factorizations with the same sum.

The 3-factorizations of 32 are $(1,1,32)$, $(1,2,16)$, $(1,4,8)$, $(2,2,8)$, and $(2,4,4)$. Their sums are 34, 19, 13, 12, and 10, no two of which are the same, so 32 does not have two different 3-factorizations with the same sum.

Among the 3-factorizations of 36 are $(1,6,6)$ and $(2,2,9)$, both of which have a sum of 13. They are different 3-factorizations, and so we have shown that 36 is the smallest positive integer that has two different 3-factorizations with the same sum.

(d) By part (b), each of the n_i needs to be the product of at least four prime numbers. The factorization of 36 is $2^2 3^2$, so we will try to generalize this.

We consider an integer of the form x^2y^2 for some integers x and y, both larger than 1. We will not assume that x and y are prime. Among the 3-factorizations of x^2y^2 are (x,x,y^2) and $(1,xy,xy)$. If their sums are equal, then $2x+y^2=1+2xy$, which can be rearranged to get $y^2-2xy+2x-1=0$ and then factored as $(y-1)(y-2x+1)=0$. From here, we can see that the equation will be satisfied if $y=2x-1$. Thus, if $y=2x-1$, then (x,x,y^2) and $(1,xy,xy)$ have the same sum. As well, as long as we assume that $x>1$, then $y>1$ as well, and these two 3-factorizations are guaranteed to be different since one of them contains 1 and the other does not.

Thus, for each integer $x>1$, the integer $x^2(2x-1)^2$ has two distinct 3-factorizations with the same sum. The table below summarizes the first few examples.

x	$x^2(2x-1)^2$	$(x, x, (2x-1)^2)$	$(1, x(2x-1), x(2x-1))$
2	36	$(2, 2, 9)$	$(1, 6, 6)$
3	225	$(3, 3, 25)$	$(1, 15, 15)$
4	784	$(4, 4, 49)$	$(1, 28, 28)$
5	2025	$(5, 5, 81)$	$(1, 45, 45)$

However, we need to do something more to get the infinite family we seek since, for example, 2025 is a multiple of 225, so $\gcd(225, 2025) = 225 \neq 1$.

To finish the problem, we will use the following fact: For any positive integer n, the integer $M = (n+1)^2(2n+1)^2$ satisfies $\gcd(n, M) = 1$.

To see this, suppose $k \geq 1$ is a factor of both n and $(n+1)^2(2n+1)^2$. This means there are integers a and b with $n = ka$ and $(n+1)^2(2n+1)^2 = 4n^4 + 12n^3 + 13n^2 + 6n + 1 = kb$. Thus,

$$\begin{aligned}
1 &= kb - (4n^4 + 12n^3 + 13n^2 + 6n) \\
&= kb - n(4n^3 + 12n^2 + 13n + 6) \\
&= kb - ka(4n^3 + 12n^2 + 13n + 6) \\
&= k[b - a(4n^3 + 12n^2 + 13n + 6)]
\end{aligned}$$

and this shows that 1 is a multiple of k because $b - a(4n^3 + 12n^2 + 13n + 6)$ is an integer. Therefore, the only possible value of k is 1, so $\gcd(n, (n+1)^2(2n+1)^2) = 1$.

This fact implies that if $k > 1$ is any divisor of n, then $\gcd(k, (n+1)^2(2n+1)^2) = 1$ since, if k and $(n+1)^2(2n+1)^2$ have a divisor greater than 1 in common, then so do n and $(n+1)^2(2n+1)^2$. We will use this to construct the infinite family of integers we seek.

We start with $n_1 = 2^2(2(2)-1)^2 = 36$, which has two different 3-factorizations with the same sum. Next, set $n_2 = (n_1+1)^2(2(n_1+1)-1)^2$, which has two distinct 3-factorizations since it is of the form $x^2(2x-1)^2$ with $x = n_1 + 1$. Expanding the second parenthetical expression, we have $n_2 = (n_1+1)^2(2n_1+1)^2$, and from the fact above, $\gcd(n_1, n_2) = 1$.

Next, set $n_3 = (n_1n_2+1)^2(2(n_1n_2+1)-1)^2 = (n_1n_2+1)^2(2n_1n_2+1)^2$. By construction, n_3 has two different 3-factorizations with the same sum. Moreover, $\gcd(n_1n_2, n_3) = 1$, so $\gcd(n_1, n_3) = \gcd(n_2, n_3) = 1$.

Continuing in this way, for each $k \geq 2$ we can define n_{k+1} from $n_1, \ldots, n_k$ by setting

$$n_{k+1} = (n_1n_2\cdots n_k + 1)^2(2n_1n_2\cdots n_k + 1)^2$$

By construction, n_{k+1} will have two different 3-factorizations with the same sum. As well, $\gcd(n_1n_2\cdots n_k, n_{k+1}) = 1$, so $\gcd(n_i, n_{k+1}) = 1$ for all $i \leq k$.

(e) We will outline two different ways to build infinite families of positive integers that have three different 3-factorizations with the same sum.

The first approach is to take any known positive integer that has three different 3-factorizations with the same sum and multiply this integer each of the perfect cubes. For example, as given in the hint, the integer 1200 has 3-factorizations $(4, 15, 20)$, $(5, 10, 24)$, and $(6, 8, 25)$, each with a sum of 39. For every positive integer n, the positive integer $1200n^3$ has 3-factorizations $(4n, 15n, 20n)$, $(5n, 10n, 24n)$, and $(6n, 8n, 25n)$. These three factorizations are all different and have the sum $39n$. Each positive integer gives a different value of $1200n^3$, so this indeed gives an infinite family of positive integers, each of which has three different 3-factorizations with the same sum.

In the second approach, for each positive integer n we define $p(n)$ to be the integer $p(n) = n(n+2)(n+4)(n+5)(n+6)(n+7)$. The integer $p(n)$ has 3-factorizations

$$\begin{aligned}
\big(n(n+7), (n+2)(n+4), (n+5)(n+6)\big) &= (n^2+7n, n^2+6n+8, n^2+11n+30) \\
\big(n(n+6), (n+2)(n+5), (n+4)(n+7)\big) &= (n^2+6n, n^2+7n+10, n^2+11n+28) \\
\big(n(n+5), (n+2)(n+7), (n+4)(n+6)\big) &= (n^2+5n, n^2+9n+14, n^2+10n+24)
\end{aligned}$$

and observe that the sums of these 3-factorizations, respectively, are

$$\begin{aligned}
(n^2+7n) + (n^2+6n+8) + (n^2+11n+30) &= 3n^2+24n+38 \\
(n^2+6n) + (n^2+7n+10) + (n^2+11n+28) &= 3n^2+24n+38 \\
(n^2+5n) + (n^2+9n+14) + (n^2+10n+24) &= 3n^2+24n+38
\end{aligned}$$

which are all the same. For example, when $n = 1$, we get $p(1) = 1\times3\times5\times6\times7\times8 = 5040$ and the 3-factorizations are $(8, 15, 42)$, $(7, 18, 40)$, and $(6, 24, 35)$.

The calculations above show that, for each n, $p(n)$ has three 3-factorizations of n with the same sum. However, there is a possibility that these 3-factorizations are not different. In fact, this problem will never occur, and as long as $n \neq 8$, not only will the given 3-factorizations of $p(n)$ be different, the nine integers occurring in them will all be different.

Consider the nine integers in the 3-factorizations. Each is a quadratic in n with a leading coefficient of 1 and none of the quadratics are the same. If an integer n has the property that two of the nine integers are the same, then we have $n^2 + an + b = n^2 + cn + d$ for some a, b, c, and d. The n^2 cancels, so in fact we have a linear equation in n. Thus, for each pair of the nine integers, there is *at most* one integer n for which those two integers are equal. For example, $n^2 + 7n = n^2 + 7n + 10$ implies $0 = 10$, so there are no integers n that will make $n^2 + 7n = n^2 + 7n + 10$ equal to each-other. As another example, if $n^2+6n+8 = n^2+10n+24$, then $-16 = 4n$, which implies $n = -4$, which is not a positive integer. In fact, of all of the 36 possible such equations, $n^2 + 7n = n^2 + 6n + 8$ is the only one with a positive integer solution, which is $n = 8$. Therefore, for every positive integer n, the given construction gives an integer that has three different 3-factorizations with the same sum.

For a hint as to how this construction was discovered, the key was to find a set of 5 positive integers with the following property: there are three different ways to choose four of the integers and break those four into two pairs so that the sum of the products of those pairs is the same. For example, the list 2, 4, 5, 6, 7 has this property because $2 \times 4 + 5 \times 6$, $2 \times 5 + 4 \times 7$, and $2 \times 7 + 4 \times 6$ are all equal to 38. Can you find another such set of five integers?

One final thought on this construction. For every integer n, either n is even or $n+5$ is even, which implies that $p(n)$ is even for all integer n. This means that $\gcd(p(n), p(m)) \geq 2$ for all positive integers m and n, so the infinite family generated in this way will not satisfy the condition in part (d). At the time of writing, we still have not found an infinite list $n_1, n_2, n_3, \ldots$ of positive integers satisfying $\gcd(n_i, n_j) = 1$ for all $i \neq j$, each of which has three different 3-factorizations with the same sum.