Problem of the Month
Problem 7: April 2022

Problem

In this problem, a 3-*factorization* of a positive integer n is a triple (a, b, c) of positive integers such that $abc = n$. Two 3-factorizations will be regarded as the same if one of them can be obtained by reordering the integers in the other. For example, $(1, 2, 3)$ and $(2, 3, 1)$ are the same 3-factorization of 6. The *sum* of the 3-factorization (a, b, c) is $a + b + c$.

(a) Suppose n has two different 3-factorizations (a, b, c) and (d, e, f) with the same sum. Prove that at most one of these 3-factorizations contains the integer 1.

(b) Suppose n has two different 3-factorizations with the same sum. Prove that n has at least four prime factors. [We allow for repetition here. For instance, $24 = 2 \times 2 \times 2 \times 3$ has four prime factors, even though it only has two *distinct* prime factors.]

(c) Find the smallest integer n that has two different 3-factorizations with the same sum.

(d) Find an infinite family $n_1, n_2, n_3, \ldots$ of positive integers satisfying

- For each i, n_i has two different 3-factorizations with the same sum.
- For each i and j, $\gcd(n_i, n_j) = 1$.

(e) Challenge: Can you find an integer with three different 3-factorizations having the same sum? Can you find infinitely many such integers? Some direction on this will be given in the hint. You may wish to try to write a computer program to get started.

Hint

(a) Assuming that $a + b + c = d + e + f$, $abc = def$, and $a = d = 1$, try to prove that either $b = e$ and $c = f$ or $b = f$ and $c = e$.

(b) Show that n cannot have two different 3-factorizations with the same sum if

   (i) n is prime.

   (ii) n is the product of two prime numbers.

   (iii) n is the product of 3 prime numbers.

   Part (a) can be used to eliminate much of the case work.

(c) The answer is less than 50. Part (b) can be used to narrow the search considerably.

(d) Use the prime factorization of the integer found in part (c) as a hint to find more integers.

(e) The smallest integer that has three different 3-factorizations with the same sum is 1200. The relevant 3-factorizations are $(4, 15, 20)$, $(5, 10, 24)$, and $(6, 8, 25)$. The next few positive integers that have three different 3-factorizations with the same sum are 1386, 1680, 1872, 2880, 2970, 3024, 3264, 3360, 3600, 3960, and 4320. These were found using a computer search, and the same computer search suggests that such integers are not especially rare (despite seeming a bit tricky to find by hand). Perhaps trying to find a pattern in the list above will lead to an infinite family or to some other interesting observations!