Problem of the Month

Problem 7: April 2022

In this problem, a 3-*factorization* of a positive integer n is a triple (a, b, c) of positive integers such that $abc = n$. Two 3-factorizations will be regarded as the same if one of them can be obtained by reordering the integers in the other. For example, $(1, 2, 3)$ and $(2, 3, 1)$ are the same 3-factorization of 6. The *sum* of the 3-factorization (a, b, c) is $a + b + c$.

(a) Suppose n has two different 3-factorizations (a, b, c) and (d, e, f) with the same sum. Prove that at most one of these 3-factorizations contains the integer 1.

(b) Suppose n has two different 3-factorizations with the same sum. Prove that n has at least four prime factors. [We allow for repetition here. For instance, $24 = 2 \times 2 \times 2 \times 3$ has four prime factors, even though it only has two *distinct* prime factors.]

(c) Find the smallest integer n that has two different 3-factorizations with the same sum.

(d) Find an infinite family $n_1, n_2, n_3, \ldots$ of positive integers satisfying

- For each i, n_i has two different 3-factorizations with the same sum.
- For each i and j, $\gcd(n_i, n_j) = 1$.

(e) Challenge: Can you find an integer with three different 3-factorizations having the same sum? Can you find infinitely many such integers? Some direction on this will be given in the hint. You may wish to try to write a computer program to get started.