Problem of the Month
Problem 6: March 2022

Problem
In this problem, we will explore the following construction: Start with the positive real number $a_1 = 1$ and an infinite sequence $m_1, m_2, m_3, \ldots$ of negative slopes that are all distinct. For $n \geq 1$, we define $a_{n+1}$ from $a_n$ as follows.

• For odd $n$, $a_{n+1}$ is the $x$-intercept of the line with slope $m_n$ through $(0, a_n)$.
• For even $n$, $a_{n+1}$ is the $y$-intercept of the line with slope $m_n$ through $(a_n, 0)$.

The diagram below illustrates this. The line through $(0, a_1)$ and $(a_2, 0)$ has slope $m_1$, the line through $(a_2, 0)$ and $(0, a_3)$ has slope $m_2$, and so on.

(a) Suppose that $m_n = -\frac{1}{2^n}$ for all $n \geq 1$.
   (i) Compute $a_2, a_3, a_4,$ and $a_5$.
   (ii) Find a general formula for $a_n$. You will likely need a separate formula for even $n$ and odd $n$. Describe what happens to $a_n$ as $n$ gets large.

(b) Suppose that $m_n = -\frac{1}{2^{\frac{n}{2}} + 1}$ for all $n$. [The exponent in the denominator is $\frac{n}{2} + 1$]
   (i) Find a general formula for $a_n$.
   (ii) Describe what happens to $a_n$ as $n$ gets large.

(c) Let $u$ and $v$ be arbitrary positive real numbers with $u \neq 1$. Give a sequence of slopes so that the sequence $a_1, a_3, a_5, a_7, \ldots$ approaches $u$ and the sequence $a_2, a_4, a_6, a_8, \ldots$ approaches $v$. Remember that the sequence of slopes should not contain any repetitions.

(d) Suppose $m_n = -\frac{1}{n}$ for all $n \geq 1$.
   (i) Find an integer $n$ so that $a_n < \frac{1}{100}$.
   (ii) Find an integer $n$ so that $a_n > 100$. 
Hint

Before attempting any of the problems, it might useful to show that $a_{n+1}$ can be expressed in terms of $a_n$ and $m_n$.

(b) In parts (i) and (ii), try computing the first few $a_n$ and looking for a pattern. Do you notice a familiar type of series forming in the exponents?

(b) If you are comfortable with logarithms, you might find that it simplifies some calculations to define $A_n = \log_2(a_n)$ and work with the $A_n$ instead. If you can find a general formula for $A_n$, then you can find a general formula for $a_n$ by using that $a_n = 2^{A_n}$.

(c) Use the idea from part (b) to construct the sequence of slopes. What happens when you change the $\frac{1}{2n} + 1$ in the exponent to $\frac{1}{2n} + c$ for some $c \neq 1$?

(d) To start, find a general formula for $a_n$. A separate formula for even $n$ and odd $n$ will probably be useful. For odd $n$, try to show that $(a_n)^2$ is less than $\frac{1}{n-1}$. Can you do something similar for even $n$?