Problem of the Month

Solution to Problem 5: February 2022

(a) After the first 100 steps are performed, a door will be open if it was toggled an odd number of times and closed if it was toggled an even number of times. Therefore, to determine whether a door is open or closed after all 100 steps, we need to determine how many times it has been toggled.

To gain intuition, we will look at a few particular doors. Door 4 is toggled in Step 1, Step 2, Step 4, and no other steps. This is because 1, 2, and 4 are the only positive factors of 4. This means Door 4 will be open after all 100 steps.

Door 10 is toggled in Step 1, Step 2, Step 5, and Step 10. No other positive integers are factors of 10, so there are no other steps in which Door 10 is toggled. Therefore, Door 10 is toggled four times, so it is closed after all 100 steps.

Recall from the problem statement that $\tau(n)$ is equal to the number of positive factors of n. Door n is toggled in Step k if and only if n is a multiple of k. Put differently Door n is toggled in Step k if and only if k is a factor of n. Thus, Door n is toggled exactly $\tau(n)$ times. Note that since $n \leq 100$ and 100 steps are performed, Step k will occur for every factor k of n.

Combining this observation with the earlier discussion, Door n will be open after all 100 steps if and only if $\tau(n)$ is odd. We will now argue that $\tau(n)$ is odd if and only if n is a perfect square.

Fix a positive integer n and suppose m is a factor of n with $1 \leq m < \sqrt{n}$. Then $\dfrac{n}{m}$ is a factor of n with the property that $\sqrt{n} < \dfrac{n}{m} \leq n$. Similarly, if m is a factor of n with the property that $\sqrt{n} < m \leq n$, then $\dfrac{n}{m}$ is a factor of n with the property that $1 \leq \dfrac{n}{m} < \sqrt{n}$. Notice that $m \times \dfrac{n}{m} = n$, so either way, if m is a positive factor of n that is not equal to $\sqrt{n}$, then $\left(m, \dfrac{n}{m}\right)$ is a factor pair for n with one factor less than $\sqrt{n}$ and the other greater than $\sqrt{n}$.

By the previous paragraph, for any positive integer n, there are an even number of positive factors of n that are different from $\sqrt{n}$. This means that the integer n has an odd number of positive factors if and only if $\sqrt{n}$ is a factor of n. If n is a perfect square, then $\sqrt{n}$ is an integer and $\sqrt{n}\sqrt{n} = n$, so $\sqrt{n}$ is a factor of n. If n is not a perfect square, then $\sqrt{n}$ is not an integer, so it cannot be a factor of n. Therefore, the integer n has an odd number of positive factors if and only if it is a perfect square. This means that Door n will be open after all 100 steps if and only if n is a perfect square.

(b) Similar to part (a), Door n gets toggled at Step d for every factor d of n. However, this time it gets toggled d times for each factor d of n. Therefore, the total number of times that a door gets toggled is equal to the sum of its positive factors.

Since a door is open after all 100 steps if and only if it has been toggled an odd number of times, we need to determine for which positive integers n the sum of the factors of n is odd.

We will consider two cases.

Case 1: Suppose n is odd. In this situation, every positive factor of n is odd. The sum of odd integers is odd if there is an odd number of them being added together and even otherwise. Thus, the positive factors of an odd integer have an odd sum if and only if there is an odd number of them. In other words, if n is odd, then Door n is open after all 100 steps if and only if $\tau(n)$ is odd. By part (a), Door n is open if and only if n is a perfect square.

Case 2: Suppose n is even. This means there is a positive integer k and an odd positive integer m such that $n = 2^k m$. The number of even factors will not affect whether the sum of the factors is odd or even since the sum of any number of even factors is always even. This means to determine if the sum of the positive factors is even or odd, we need only consider the odd factors. If there is an odd number of odd factors, then the sum of the positive factors will be odd. Otherwise, it will be even.

If d is an odd factor of $n = 2^k m$, then d must be a factor of m. Conversely, if d is a factor of m, then d is an odd factor of n. Thus, the number of odd factors of $n = 2^k m$ is equal to the number of odd factors of m. We are assuming that m is odd, so it has an odd number of positive factors if and only if it is a perfect square (by the previous case). Therefore, n has an odd number of odd factors if and only if m is a perfect square.

Putting the cases together, we have that every positive integer n can be written in the form $2^k m$ where k is a *non-negative* integer and m is an odd positive integer. Door n will be open after all 100 steps if and only if m is a perfect square.

Thus, Door n will be open after all 100 steps if and only if n is the product of a power of 2 and an odd perfect square. This description can be simplified even more. Suppose $n = 2^k m$ where k is non-negative and m is an odd perfect square. This means $m = r^2$ for some r. If k is even, then $n = 2^k m = 2^k r^2 = \left(2^{\frac{k}{2}} r\right)^2$, so n itself is a perfect square. If k is odd, then $k - 1$ is even, so $n = 2^k m = 2\left(2^{k-1} r^2\right) = 2\left(2^{\frac{k-1}{2}} r\right)^2$, so n is two times a perfect square.

We can now say that Door n will be open after all 100 steps if and only if n is a perfect square or n is two times a perfect square.

(c) For this part, we still need to identify whether Door n, after n steps, has been toggled an even or an odd number of times. It will be open if and only if it has been toggled an odd number of times.

In the first n steps, Door n gets toggled $\tau(d)$ times for every factor d of n. From part (a), we know that $\tau(d)$ is even unless d is a perfect square. By reasoning we have used earlier, this means that Door n will be open after n steps if and only if an odd number of perfect squares divide n.

Every positive integer n can be written uniquely in the form $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ where the p_i are distinct prime numbers and the e_i are positive integers. A positive integer d is a factor of n if and only if $d = p_1^{f_1} p_2^{f_2} \cdots p_k^{f_k}$ for some integers f_i with $0 \leq f_i \leq e_i$ for each i. For d to be a perfect square, each of the f_i must be even. To count the number of factors of

n that are perfect squares, we can count how many even integers there are from 0 to e_i inclusive for each i, then take the product of these values. This is because we obtain a perfect square factor for every choice of even integers f_i, and the choices are independent.

The product of positive integers is odd if and only if all of the integers being multiplied together are odd. Thus, for $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ to have an odd number of perfect-square factors, it must be true that for each i there is an odd number of even integers between 0 and e_i inclusive. For a fixed i, we consider four cases. In each case, we will determine how many even positive integers there are between 0 to e_i inclusive.

Case 1: $e_i = 4r$ for some integer r. In this case, $0, 2, 4, \ldots, 4r-2, 4r$ are the even integers from 0 to e_i inclusive. There are $\frac{4r}{2} + 1 = 2r + 1$ of them, so in this case, there is an odd number of choices for f_i.

Case 2: $e_i = 4r + 1$ for some integer r. In this case, $0, 2, 4, \ldots, 4r - 2, 4r$ are the even integers from 0 to e_i inclusive. By the same calculation as Case 1 above, there is an odd number of choices for f_i.

Case 3: $e_i = 4r+2$ for some integer r. In this case, the even integers from 0 to e_i inclusive are $0, 2, 4, \ldots, 4r, 4r + 2$. There are $2r + 2$ integers in this list, so there is an even number of choices for f_i.

Case 4: $e_i = 4r+3$ for some integer r. In this case, the even integers from 0 to e_i inclusive are the same as those in Case 3, so there is again an even number of choices for f_i.

By the reasoning given before we considered the cases, we get that Door n will be open after n steps if and only if each e_i is either a multiple of 4 or one more than a multiple of 4.

For the given integer $K = 2^9 3^4 5^{13} 7^{12}$, each exponent has this property, so Door K will be open after K steps.