Problem of the Month
Problem 5: February 2022

In each part of this problem, there is a hallway containing of \( K \) doors numbered consecutively from 1 to \( K \) that are all initially closed. To \textit{toggle} a door means to open it if it is closed and to close it if it is open. We will also use the notation that for a positive integer \( n \), \( \tau(n) \) is equal to the number of positive integer factors of \( n \). For example, \( \tau(1) = 1 \) since 1 has exactly one positive factor and for a prime number \( p \), we always have \( \tau(p) = 2 \) since prime numbers have exactly two positive factors. For another example \( \tau(10) = 4 \) since it has four positive integer factors, 1, 2, 5, and 10.

(a) In this part, \( K = 100 \). 100 “steps” are performed as follows:

- In step 1, every door that is numbered with a multiple of 1 is toggled.
- In step 2, every door that is numbered with a multiple of 2 is toggled.
- In step 3, every door that is numbered with a multiple of 3 is toggled.

In step \( n \), every door that is numbered with a multiple of \( n \) is toggled. After all 100 steps are performed, which doors are open?

(b) In this part, \( K = 100 \). As with part (a), 100 steps are performed with one step for each integer \( n \) from 1 through \( K \). This time, in step \( n \), each door that is numbered with a multiple of \( n \) is toggled \( n \) times. For example, in step 5, each door that is numbered with a multiple of 5 is to be toggled 5 times. After all 100 steps are performed, which doors are open?

(c) In this part, \( K = 2^9 \times 3^4 \times 5^{13} \times 7^{12} \). As with parts (a) and (b), a step is performed for each positive integer \( n \) from 1 through \( K \). In step \( n \), every door that is numbered by a multiple of \( n \) is toggled \( \tau(n) \) times. For example, in step 5, every door that is numbered by a multiple of 5 is toggled \( \tau(5) = 2 \) times.

After all \( K \) steps are performed, is the door numbered with \( K \) open or closed?