Problem of the Month

Solution to Problem 4: January 2022

Before starting the solution, we include a brief discussion on how the roots of polynomials are related to their coefficients.

Consider the quadratic polynomial $ax^2 + bx + c$ where a, b, and c are real numbers with $a \neq 0$. If u and v are the roots of the polynomial, then $u + v = -\frac{b}{a}$ and $uv = \frac{c}{a}$. This is because $ax^2 + bx + c$ has the same roots as $x^2 + \frac{b}{a}x + \frac{c}{a}$, and since u and v are the roots, we must have

$$x^2 + \frac{b}{a}x + \frac{c}{a} = (x - u)(x - v) = x^2 - (u + v)x + uv.$$

By similar reasoning, if $x^3 + ax^2 + bx + c$ has roots u, v, and w, then it must factor as

$$x^3 + ax^2 + bx + c = (x - u)(x - v)(x - w)$$

and expanding, we find that $-a = u + v + w$, $b = uv + vw + wu$, and $c = -uvw$. These are often known as (some of) *Vieta's formulas* and they are very useful when studying the roots of polynomials.

(a) Let r be the common root. Then $2r^2 - 1275r + 194292 = 0$ and $r^2 - 635r + 96516 = 0$. Doubling the second equation gives $2r^2 - 1270r + 193032 = 0$. Subtracting the equation $2r^2 - 1275r + 194292 = 0$ from the equation $2r^2 - 1270r + 193032 = 0$ gives

$$\begin{aligned} 0 &= 0 - 0 \\ &= (2r^2 - 1270r + 193032) - (2r^2 - 1275r + 194292) \\ &= 5r + 193032 - 194292 \\ &= 5r - 1260 \end{aligned}$$

which implies $5r = 1260$. Solving for r gives $r = 252$. From the discussion before the solution, the sum of the roots of $x^2 - 635x + 96516$ is 635, so the other root is $635 - 252 = 383$. Similarly, the sum of the roots of $2x^2 - 1275x + 194292$ is $\frac{1275}{2}$ and one of the roots is 252, so the other is $\frac{1275}{2} - \frac{504}{2} = \frac{771}{2}$.

(b) Suppose r is a root of $p(x)$.

We will first assume r is a repeated root of $p(x)$ and deduce that r is a root of $q(x)$. Since r is a repeated root of $p(x)$, $(x - r)^2$ divides evenly into $p(x)$. This means there must be some other root t such that $(x - r)^2(x - t) = x^3 + ax^2 + bx + c$. From the formulas before the solution, we have that $a = -2r - t$, $b = r^2 + 2rt$, and $c = -r^2t$. Thus

$$\begin{aligned} q(r) &= 3r^2 + 2ar + b \\ &= 3r^2 + 2(-2r - t)r + r^2 + 2rt \\ &= 3r^2 - 4r^2 - 2rt + r^2 + 2rt \\ &= 0 \end{aligned}$$

and so r is a root of $q(x)$. Thus, if r is a repeated root of $p(x)$, then r is a root of $q(x)$.

Now we will assume r is a root of $q(x)$ and deduce that $(x-r)^2$ divides evenly into $p(x)$. Since r is a root of $q(x)$, we have $3r^2 + 2ar + b = 0$, so

$$b = -(3r^2 + 2ar) \tag{1}$$

As well, we are still assuming that $p(r) = 0$, which means $r^3 + ar^2 + br + c = 0$ or $c = -r^3 - ar^2 - br$. Multiplying $3r^2 + 2ar + b = 0$ through by r and rearranging gives $-br = 3r^3 + 2ar^2$, and substituting this into $c = -r^3 - ar^2 - br$ gives

$$\begin{aligned} c &= -r^3 - ar^2 + (3r^3 + 2ar^2) \\ &= 2r^3 + ar^2 \end{aligned} \tag{2}$$

Using these equations, we have

$$\begin{aligned} (x-r)^2(x+a+2r) &= (x^2 - 2rx + r^2)(x+a+2r) \\ &= x^3 + (a + 2r - 2r)x^2 + (r^2 - 2ar - 4r^2)x + (ar^2 + 2r^3) \\ &= x^3 + ax^2 - (3r^2 + 2ar)x + (2r^3 + ar^2) \\ &= x^3 + ax^2 + bx + c && \text{(1) and (2)} \\ &= p(x) \end{aligned}$$

This shows that if r is a root of $p(x)$ and $q(x)$, then $(x-r)^2 = x^2 - 2rx + r^2$ divides evenly into $p(x)$, which means r is a repeated root of $p(x)$.

Here is another solution that uses the following fact: If

$$p(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

is a polynomial of degree n with real coefficients, then there are *complex* numbers $r_1, \ldots, r_n$ such that $p(x) = (x - r_1)(x - r_2)\cdots(x - r_n)$. This is a formulation of the famous *Fundamental Theorem of Algebra*. Proving this theorem is far beyond the scope of this activity, but to readers who recognize $q(x)$ as the *derivative* of $p(x)$ and who know the *product rule*, it offers a more enlightening proof of the fact in this problem.

Suppose $p(x) = (x - r_1)(x - r_2)(x - r_3)$. Observe that $-(r_1 + r_2 + r_3) = a$ as well as $r_1r_2 + r_1r_3 + r_2r_3 = b$ by Vieta's formulas, which will still hold for complex numbers. Interestingly, even though the roots r_1, r_2, and r_3 may not be real, their sum and the sum of their pairwise products will be real. For reasons that may seem utterly mysterious unless you have seen the product rule, we have the following:

$$\begin{aligned} q(x) &= 3x^2 + 2ax^2 + b \\ &= 3x^2 - 2(r_1 + r_2 + r_3)x + r_1r_2 + r_1r_3 + r_2r_3 \\ &= 3x^2 - \big((r_1 + r_2) + (r_2 + r_3) + (r_3 + r_1)\big)x + r_1r_2 + r_2r_3 + r_3r_1 \\ &= \big(x^2 - (r_1 + r_2)x + r_1r_2\big) + \big(x^2 - (r_2 + r_3)x + r_2r_3\big) + \big(x^2 - (r_3 + r_1)x + r_3r_1\big) \\ &= (x - r_1)(x - r_2) + (x - r_2)(x - r_3) + (x - r_3)(x - r_1) \end{aligned}$$

and from this, the solution falls out almost immediately. The roots of $p(x)$ are r_1, r_2, and r_3. Can you see why this form of $q(x)$ implies that $q(x)$ and $p(x)$ share a root exactly when at least two of r_1, r_2, and r_3 are the same?

(c) Set $D = (u-v)^2(v-w)^2(w-u)^2$. Since u, v, and w are the roots of $x^2 + bx + c$ (note that $a = 0$), Vieta's formulas imply the equations

$$u + v + w = 0 \tag{1}$$
$$uv + vw + wu = b \tag{2}$$
$$uvw = -c \tag{3}$$

Adding $3uv$ to both sides of (2) and factoring gives $b + 3uv = 4uv + w(u+v)$. From (1), we also have that $w = -u - v$. Substituting this into $b + 3uv = 4uv + w(u+v)$ gives

$$\begin{aligned}
b + 3uv &= 4uv + w(u+v) \\
&= 4uv - (u+v)(u+v) \\
&= 4uv - u^2 - 2uv - v^2 \\
&= -(u^2 - 2uv + v^2) \\
&= -(u-v)^2
\end{aligned}$$

from which it follows that $-b - 3uv = (u-v)^2$. A very similar calculation shows that $-b - 3vw = (v-w)^2$ and $-b - 3wu = (w-u)^2$. Using these three equations, we have

$$\begin{aligned}
D &= (u-v)^2(v-w)^2(w-u)^2 \\
&= (-3uv - b)(-3vw - b)(-3wu - b) \\
&= -(b + 3uv)(b + 3vw)(b + 3wu) \\
&= -\big(b^3 + 3b^2(uv + vw + wu) + 9b(u^2vw + uv^2w + uvw^2) + 27u^2v^2w^2\big) \\
&= -\big(b^3 + 3b^2(b) + 9buvw(u + v + w) + 27(uvw)^2\big) && (2) \\
&= -\big(4b^3 + 9buvw(0) + 27(-c)^2\big) && (1) \text{ and } (3) \\
&= -4b^3 - 27c^2
\end{aligned}$$

(d) If we were to expand

$$p\left(x - \frac{a}{3}\right) = \left(x - \frac{a}{3}\right)^3 + a\left(x - \frac{a}{3}\right)^2 + b\left(x - \frac{a}{3}\right) + c,$$

the x^2 term must come from $\left(x - \frac{a}{3}\right)^3$ and $a\left(x - \frac{a}{3}\right)^2$. The x^2 term coming from $\left(x - \frac{a}{3}\right)^3$ is $-3\left(\frac{a}{3}x^2\right) = -ax^2$ and the x^2 term coming from $a\left(x - \frac{a}{3}\right)^2$ is ax^2. Their sum is 0, so the coefficient of x^2 in the polynomial $p\left(x - \frac{a}{3}\right)$ must be 0.

Suppose r is a root of $q(x)$. Then $q(r) = 0$, which means $p\left(r - \frac{a}{3}\right) = 0$. This means $r - \frac{a}{3}$ is a root of $p(x)$. As well, if t is a root of $p(x)$, then $t + \frac{a}{3}$ is a root of $q(x)$ since

$$q\left(t + \frac{a}{3}\right) = p\left(t + \frac{a}{3} - \frac{a}{3}\right) = p(t) = 0$$

Thus, the roots of $p(x)$ are exactly the roots of $q(x)$ with $\frac{a}{3}$ subtracted from them. If you have studied horizontal translations of functions, you may notice that $q(x)$ is a horizontal translation of $p(x)$ to the right by $\frac{a}{3}$, which gives a geometric explanation of why the roots of one polynomial are just the roots of the other after a horizontal shift.

This may not seem like an important observation, but it is of both algebraic and historical significance. In a theoretical sense, it tells us that if we can understand the roots of cubics without a quadratic term, then we can understand the roots of *every* cubic. Just like there is a "quadratic formula" that will produce the exact roots of any quadratic polynomial in terms of its coefficients, there is a "cubic formula". The general cubic formula is quite complicated, but its specialized version in the case where the quadratic term is missing is much simpler.

In fact, this specialized formula was discovered many years before the general formula was, which may seem surprising since, as observed in this question, the only barrier between the two appears to be a simple translation. Indeed, the observation in this problem is the one found by Cardano that finally generalized the specialized cubic formula to handle more general cubics. By today's standards, the observation is as simple as it seems. However, one must keep in mind that the techniques of modern algebra were not available in the 16^{th} century. What seems today like a quick algebraic substitution and manipulation was a geometric observation that, by today's standards, would seem contrived and unnecessary.

Cardano's formula will find the roots of any cubic in terms of its coefficients. However, because of the need to extract roots of negative and sometimes complex numbers, mathematicians of the day, in a sense, did not know how to use the formula to its full potential. You might wish to do some research on the history of the cubic formula.

(e) We will consider the polynomial

$$q(x) = p\left(x - \frac{a}{3}\right) = p\left(x - \frac{-135}{3}\right) = p(x+45)$$

We will not show the calculations, but it it can be checked that

$$(x+45)^3 - 135(x+45)^2 + 5832(x+45) - 81648 = x^3 - 243x - 1458$$

and so we will find the roots of $x^3 - 243x - 1458$ and then subtract -45 from them to get the roots of the given polynomial.

To find the roots of this simpler cubic, we first calculate its discriminant. Here, $b = -243$ and $c = -1458$. So

$$-4b^3 - 27c^2 = -4(-243)^3 - 27(-1458)^2 = 57395628 - 57395628 = 0$$

and by part (c), the polynomial $x^3 - 243x - 1458$ must have a repeated root.

By part (b), that repeated root, r, must also be a root of the quadratic $3x^2 - 243$. If $3r^2 - 243 = 0$, then $r^2 - 81 = 0$, which means $r = \pm 9$. [At this point, we could test both possibilities to see which is the root of the cubic of interest, but we will go through the calculation to demonstrate how this sort of technique works more generally.] Since $r^2 - 81 = 0$, $r^3 - 81r = 0$. We also have that $r^3 - 243r - 1458 = 0$. Subtracting this equation from $r^3 - 81r = 0$ gives $162r + 1458 = 0$, so $r = -\frac{1458}{162} = -9$.

We now have that -9 is a root of $x^3 - 243x - 1458$, and in fact, it must be a repeated root. This means we can factor $(x+9)^2$ out of the polynomial. Indeed, a bit of polynomial division reveals that

$$x^3 - 243x - 1458 = (x^2 + 18x + 81)(x - 18)$$

This means the final root of $x^3 - 243x - 1458$ is 18, so the roots of the original polynomial are $-9 + 45 = 36$, which is a repeated root, and $18 + 45 = 63$. You can check that the polynomial in the problem factors as

$$(x - 36)(x - 36)(x - 63) = x^3 - 135x^2 + 5832x - 81648$$

While it may have been just as well to use some other technique like the rational roots theorem, the point being made here is that there are potentially a few simplifying tricks when it comes to finding roots of polynomials. The first is the substitution at the beginning to change our focus to a polynomial with one of its coefficients equal to 0 (can you see how this is done for polynomials of higher degree?). The next is to check the discriminant to detect repeated roots. If there is a repeated root, then, in principle, it is easier to find than a non-repeated root because of the trick involving the *derivative* (this also generalizes to higher degree polynomials). There are many algorithms known for factoring polynomials, especially those with integer coefficients. They are surprisingly efficient because of tricks like these.