Problem of the Month

Problem 4: January 2022

The goal of this problem is to work through some techniques that can sometimes help find the roots of polynomials. The statements of some parts of this problem refer to *repeated roots*, which we will now define. Suppose r is a root of the polynomial $p(x)$, that is, $p(r) = 0$. You may already know that if $p(r) = 0$, then $(x - r)$ divides evenly into $p(x)$. We say that r is a repeated root of $p(x)$ if $(x - r)^2$ divides evenly into $p(x)$. For example, 1 is a repeated root of $x^2 - 2x + 1$ because $x^2 - 2x + 1 = (x - 1)^2$, and 2 is a repeated root of $x^4 - 5x^3 + 6x^2 + 4x - 8$ since $x^4 - 5x^3 + 6x^2 + 4x - 8 = (x - 2)^2(x^2 - x - 2)$.

(a) The polynomials $p(x) = 2x^2 - 1275x + 194292$ and $q(x) = x^2 - 635x + 96516$ have a root in common. Determine both roots of both polynomials without using the quadratic formula.

(b) Let $p(x) = x^3 + ax^2 + bx + c$ be a polynomial with a root r. Show that r is a repeated root of $p(x)$ if and only if r is a root of the polynomial $q(x) = 3x^2 + 2ax + b$.

You may recognize $q(x)$ as the *derivative* of $p(x)$. If you are familiar with derivatives, you might want to try to generalize this part.

(c) Suppose $p(x) = x^3 + bx + c$ has roots u, v, and w (which may not all be different). Express the quantity $(u - v)^2(v - w)^2(w - u)^2$ in terms of b and c. This quantity is known as the discriminant of $p(x)$, and this exercise shows that its value can be determined from the coefficients without knowing the roots. Explain how, without knowing any of the roots, it is possible to determine if a cubic of the form $x^3 + bx + c$ has a repeated root.

(d) Consider the polynomial $p(x) = x^3 + ax^2 + bx + c$. Show that the coefficient of x^2 in the polynomial $q(x) = p\left(x - \frac{a}{3}\right)$ is equal to 0. Explain how the roots of $p(x)$ can be found easily if the roots of $q(x)$ are known.

(e) Find all roots of the polynomial $p(x) = x^3 - 135x^2 + 5832x - 81648$.