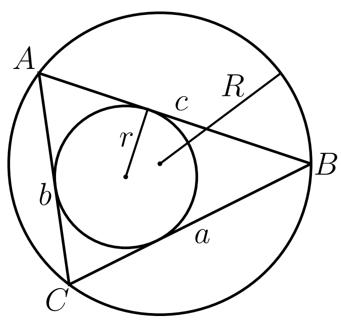
Problem of the Month

Problem 3: December 2021

Before stating the problem, we will introduce some notation and terminology.

- In $\triangle ABC$, we will denote the length of BC by a, the length of AC by b, and the length of AB by c.
- The *semiperimeter* of $\triangle ABC$ will be denoted by s and is equal to $\dfrac{a+b+c}{2}$.
- The *incircle* of $\triangle ABC$ is the unique circle that is tangent to all three sides of $\triangle ABC$. Its radius is called the *inradius* of $\triangle ABC$ and is denoted by r. An important fact about the incircle is that its centre is at the intersection of the three angle bisectors of the triangle.
- The *circumcircle* of $\triangle ABC$ is the unique circle on which all three of A, B, and C lie. Its radius is called the *circumradius* of $\triangle ABC$ and is denoted by R. An important fact about the circumcircle is that its centre is at the intersection of the perpendicular bisectors of the three sides of the triangle.

The diagram below illustrates some of the information above.



This problem is about right-angled triangles. Most of us are aware of the famous Pythagorean theorem, but there are other interesting properties only satisfied by right-angled triangles.

(a) Suppose $\triangle ABC$ is right-angled at C and that h is the length of the altitude from C to AB. Show that $\dfrac{1}{a^2}+\dfrac{1}{b^2}=\dfrac{1}{h^2}$.

(b) Suppose $\triangle ABC$ is right-angled. Show that $\cos^2\angle A+\cos^2\angle B+\cos^2\angle C=1$.

(c) Suppose $\triangle ABC$ is right-angled. Show that $s=r+2R$.

(d) Suppose $\triangle ABC$ satisfies $a^2+b^2=c^2$. Prove that $\angle C=90^\circ$.

(e) Suppose $\triangle ABC$ satisfies $\cos^2\angle A+\cos^2\angle B+\cos^2\angle C=1$. Prove that $\triangle ABC$ is right-angled.

(f) Suppose $\triangle ABC$ satisfies $s=r+2R$. Show that $\triangle ABC$ is right-angled. [A solution to this problem will likely require some general identities involving the inradius and circumradius. Some specific useful identities will be given in the hint.]