Problem of the Month

Problem 1: October 2021

Problem

Suppose a, b, and c are positive integers. In this problem, a *non-negative solution* to the equation $ax + by = c$ is a pair $(x, y) = (u, v)$ of integers with $u \geq 0$ and $v \geq 0$ satisfying $au + bv = c$. For example, $(x, y) = (7, 0)$ and $(x, y) = (3, 3)$ are non-negative solutions to $3x + 4y = 21$, but $(x, y) = (-1, 6)$ is not.

(a) Determine all non-negative solutions to $5x + 8y = 120$.

(b) Determine the largest positive integer c with the property that there is no non-negative solution to $5x + 8y = c$.

In parts (c), (d), and (e), a and b are assumed to be positive integers satisfying $\gcd(a, b) = 1$.

(c) Determine the largest non-negative integer c with the property that there is no non-negative solution to $ax + by = c$. The value of c should be expressed in terms of a and b.

(d) Determine the number of non-negative integers c for which there are exactly 2021 non-negative solutions to $ax + by = c$. As with part (c), the answer should be expressed in terms of a and b.

(e) Suppose $n \geq 1$ is an integer. Determine the sum of all non-negative integers c for which there are exactly n nonnegative solutions to $ax + by = c$. The answer should be expressed in terms of a, b, and n.

Fact: You may find it useful that for integers a and b with $\gcd(a, b) = 1$, there always exist integers x and y such that $ax + by = 1$, though x and y may not be non-negative.

Hint

(a) An exhaustive search is a reasonable approach to this problem. It can be made easier if you notice that x must be a multiple of 8 and that y must be a multiple of 5.

(b) Find a positive integer c with the property that $ax + by = c$, $ax + by = c + 1$, $ax + by = c + 2$, $ax + by = c + 3$, and $ax + by = c + 4$ all have non-negative solutions.

(c), (d), (e) As always, it is good to work out a few small examples to try to guess a pattern. It might be useful to understand the set of *all* integer solutions to $ax + by = c$ for fixed a, b, and c with $\gcd(a, b) = 1$. Once you do this, you might consider the integer solution $(x, y) = (u, v)$ with u negative but as close to 0 as possible.